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PRK, a Constructive Classical Logic

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PRK, UNA LÓGICA CONSTRUCTIVA CLÁSICA

Esta tesis presenta el sistema PRK, un sistema lógico en el cual las nociones de prueba y refutación son duales. Este sistema extiende a la lógica clásica y es constructivo en el sentido de que se lo puede dotar de una interpretación computacional con buenas propiedades. Las fórmulas de PRK se clasifican a lo largo de dos ejes, dependiendo de su *positividad* (afirmación o negación) y su *fuerza* (fuerte o clásica). Las proposiciones fuertes se demuestran, canónicamente, con reglas de introducción, mientras que las proposiciones clásicas se demuestran por reducción al absurdo.

El sistema PRK resulta ser correcto y completo con respecto a una clase de modelos de Kripke, definida en este mismo trabajo. Siguiendo la correspondencia de Curry–Howard, se formaliza un cálculo asociado a PRK, denominado λ^{PRK}, cuyo sistema de tipos se corresponde con la lógica PRK. Se establecen varias propiedades sobre λ^{PRK}, incluyendo preservación de tipos, confluencia y una caracterización de las formas normales de las pruebas y refutaciones. La terminación fuerte del cálculo λ^{PRK} se demuestra a través de una traducción a System F extendido con ecuaciones recursivas entre tipos, y apoyándose en un resultado de Mendler.

Por último, se considera una extensión a segundo orden del sistema PRK, junto con el cálculo correspondiente λ_2^{PRK}. Se extienden a este marco los resultados anteriormente mencionados, exceptuando la terminación fuerte de λ_2^{PRK}, que queda abierta como trabajo futuro.

Palabras clave: Lógica, Curry–Howard, Lógica Clásica, Proposiciones como Tipos, Lógica Constructiva, Semántica de Kripke.

PRK, A CONSTRUCTIVE CLASSICAL LOGIC

This thesis introduces PRK, a constructive classical logic with dual proofs and refutations that refines classical logic and provides a well behaved computational interpretation for it. Formulas in PRK can be classified along two axes, depending on their *positivity* (affirmation or denial) and their *strength* (strong or classical). Strong propositions are, canonically, proved with introduction rules, whereas the proof of a classical proposition always proceeds by contradiction.

The system PRK is shown to be sound and complete with respect to a particular kind of Kripke semantics, also defined in this work. A calculus for PRK, dubbed λ^{PRK}, is formalized. Its type system is in close correspondence with the logical rules of PRK, in the sense of the propositions-as-types paradigm. A number of properties, including subject reduction, confluence, and a characterization of canonical proofs and refutations, are established. Strong normalization of this calculus is proved via a translation to System F with Mendler-style recursive type constraints.

Finally, an extension of PRK to second order logic is presented, including a corresponding calculus λ_2^{PRK}. The aforementioned results are extended to this setting, except for strong normalization of λ_2^{PRK}, which is left as future work.

Keywords: Logic, Curry–Howard, Classical Logic, Propositions as Types, Constructive Logic, Kripke Semantics.

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CONTENTS

1. INTRODUCTION

The propositions-as-types correspondence, also known as the Curry–Howard isomorphism, creates a bridge between two, close, yet distinct, scientific fields: it tells us that computer programs and mathematical proofs are just two faces of the same coin. Under this correspondence, Logic and Computer Science are able to feed each other with questions, techniques, and results. For example, the development of linear logic, a substructural logic, in the 1980s, unleashed the development of linear type systems as a possible solution for resource management in practical programming languages. On the opposite direction, the formalization of computing as the transformation of terms through syntactic rules created a new framework for research in proof theory, giving new life to old concerns, such as the question of when two proofs should be considered equal.

This bridge, however, did not connect logics and type systems exactly as it may have been expected. The λ-calculus, the computational formalism on which the Curry-Howard correspondence was first developed, does not turn out to provide a correspondence between terms and proofs in *classical logic*, the "standard" logic used by most mathematicians, but rather with proofs in *intuitionistic logic*, a constructive logic, slightly more restrictive than classical logic[1]. Therefore, almost instantly after its discovery, this connection raised an obvious question without a trivial answer: is it possible to provide a computational interpretation for classical logic?

Griffin [1] was the first to propose an answer, by observing that Felleisen's $\mathcal{C}$ control operator, related to the `call/cc` operator in the Scheme family of programming languages, can be given the type $\neg\neg A \to A$. From the logical point of view, this corresponds to the principle of *double negation elimination*, which is classically valid, but not intuitionistically so. Griffin's proposal had, however, some shortcomings; for instance, its reduction rules were tied to a particular evaluation strategy, and he had to consider the type of a program to be $\bot$ (*i.e.* the type without inhabitants), otherwise reduction wouldn't preserve typing.

A couple of years later, Parigot [2] managed to overcome some of these problems and introduced the $\lambda\mu$-calculus, a calculus whose propositions-as-types counterpart is classical natural deduction, a natural deduction system with multiple conclusions, that allows to prove classically valid formulas. Following Parigot's work, many others have achieved to find satisfactory computational interpretations of classical logic (see for instance [3, 4]).

This area of research has two obvious applications. First, it could allow us to obtain mathematical descriptions of programming languages with control flow operators, such as jumps and exceptions, without the need of translating them into a purely functional framework as the λ-calculus [5]. Second, it could lay the foundations for implementing theorem provers based on classical logic, without giving up the ability to compute inside the system.

In this thesis we introduce a logical system called PRK. Then, we derive a corresponding calculus, dubbed λ^{PRK} and we show that it serves as a well behaved computational interpretation for classical logic.

[1] Intuitionistic logic differs from classical logic in that it does not assume the principle of excluded middle, namely $A \vee \neg A$, as an axiom.

The PRK *system* It is well known that intuitionistic logic enjoys certain properties that make it a perfect candidate to be used as a type system for programming languages, with respect to the propositions-as-types correspondence; for example, the disjunctive property tells us that the proof of a disjunction, $A \vee B$, can always be normalized to a proof of either A or B, and that this proof can effectively be extracted. On the other hand, classical logic can't guarantee a similar property, which is the reason why $A \vee \neg A$ is classically, but not intuitionistically, valid.

However, the strong requirements imposed by intuitionistic logic do not compose well with negation, since, even if a proof of a disjunction can be used to extract a proof of one of the disjuncts, a refutation of a conjunction ($\neg(A \wedge B)$) cannot be given the same treatment, that is, it cannot be used to extract a refutation of one of the conjuncts. This is because negation, in intuitionistic logic, is encoded via contradiction ($\neg A \equiv (A \to \bot)$), and De Morgan's laws are not generally valid ($\neg(A \wedge B) \not\equiv \neg A \vee \neg B$)

Nelson [6] proposed a solution to recover the symmetry between proof and refutation, named *constructible falsity*. His approach can be summarized as decorating formulas with a positivity marker at the top level, for example, the formula A can be decorated either positively, A^+, or negatively, A^-. This system can be summarized by the equations below. Note that the duality between proof and refutation indeed reappears, for example, the set of proofs of a conjunction can be understood as the cartesian product of the sets of proofs of the conjuncts, while the set of refutations of a conjunction is the coproduct of the sets of refutations of the conjuncts:

$$\begin{array}{rclrcl} (A \wedge B)^+ & \approx & A^+ \times B^+ & (A \wedge B)^- & \approx & A^- \uplus B^- \\ (A \vee B)^+ & \approx & A^+ \uplus B^+ & (A \vee B)^- & \approx & A^- \times B^- \\ (\neg A)^+ & \approx & A^- & (\neg A)^- & \approx & A^+ \end{array}$$

The system PRK presented in this thesis can be seen as an extension of Nelson's system, where an extra axis for distinguishing formulas is introduced. Formulas are not only divided by their positivity (affirmation/denial), but also by their strength (strong/classical), as can be seen below.

	affirmation	denial
strong	A^+	A^-
classical	$A^\oplus$	$A^\ominus$

These two axes present a logic where a formula A can be moded in four different ways, written A^+ (strong affirmation), $A^\oplus$ (classical affirmation), A^- (strong denial), and $A^\ominus$ (classical denial). The distinction between affirmations and denials follows the ideas by Nelson, thus introducing the duality lost in intuitionistic logic discussed above. The strength axis makes a distinction between strong propositions (A^+, A^-) and classical propositions ($A^\oplus$, $A^\ominus$). Proofs of strong propositions must be constructive in the sense that they must (canonically) be built using an introduction rule of the corresponding connective, while proofs of classical propositions always proceed by *reductio ad absurdum*.

The interpretation of these new set of decorated formulas follows Nelson's ideas, but with some new cases modelling the interaction between strong and classical formulas:

$$\begin{array}{rclrcl} (A \wedge B)^+ & \approx & A^\oplus \times B^\oplus & (A \wedge B)^- & \approx & A^\ominus \uplus B^\ominus \\ (A \vee B)^+ & \approx & A^\oplus \uplus B^\oplus & (A \vee B)^- & \approx & A^\ominus \times B^\ominus \\ (\neg A)^+ & \approx & A^\ominus & (\neg A)^- & \approx & A^\oplus \\ A^\oplus & \approx & A^\ominus \to A^+ & A^\ominus & \approx & A^\oplus \to A^- \end{array} \tag{1.1}$$

Contributions The contributions of this thesis can be summarized as follows:

- we define the PRK system;
- we give it semantic meaning via Kripke models;
- we present a typed λ-calculus based on PRK, called λ^{PRK}, and prove that it enjoys desired properties;
- we show how PRK relates to classical logic, and provide a computational interpretation for classical logic through λ^{PRK};
- we set the field up for the future study of a second order version of PRK.

Some of the results of this thesis were previously published on the 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS) [7].

Structure This thesis is structured in multiple chapters. Chapter 2 presents PRK as a logical system, and proves some of its fundamental properties. Chapter 3 defines a notion of Kripke semantics for PRK, and shows soundness and completeness of PRK with respect to this notion of semantics. Chapter 4 introduces a typed λ-calculus based on PRK including reduction rules, and it shows that it enjoys properties such as subject reduction, confluence, and termination. Chapter 5 relates PRK with classical logic. Chapter 6 extends PRK for second order logic. Finally, Chapter 7 concludes and proposes some possible lines of future work.

2. THE LOGICAL SYSTEM PRK

In this section we define the PRK logic in natural deduction style. The rules will follow directly from the equations (1.1) discussed in the introduction.

Following the definition, we show some basic properties and admissible rules of our system. These results will be the main examples used, throughout this thesis, to show how the PRK system presents itself in its different forms, so it is worth understanding them thoroughly.

1 System Definition

Given a denumerable set of *propositional variables* $\alpha, \beta, \gamma, \ldots$. The set of *pure propositions* is given by the abstract syntax:

$$\begin{array}{rcll} A, B, C, \ldots & ::= & \alpha & \text{propositional variable} \\ & | & A \wedge B & \text{conjunction} \\ & | & A \vee B & \text{disjunction} \\ & | & \neg A & \text{negation} \end{array}$$

As mentioned in the introduction, propositions can be classified along two dimensions, obtaining *moded propositions* (or just *propositions*), given by:

$$\begin{array}{rcll} P, Q, R, \ldots ::= & ::= & A^{+} & \text{strong affirmation} \\ & | & A^{-} & \text{strong denial} \\ & | & A^{\oplus} & \text{classical afirmation} \\ & | & A^{\ominus} & \text{classical denial} \end{array}$$

The first of these dimensions is called *sign*, and distinguishes between *affirmations* (A^{+} and $A^{\oplus}$) and *denials* (A^{-} and $A^{\ominus}$), sometimes also called *positive* and *negative* propositions. The second dimension is called *strength*, and distinguishes between *strong propositions* (A^{+} and A^{-}) and *weak propositions* ($A^{\oplus}$ and $A^{\ominus}$) (also called *classical propositions*). Note that modes cannot be nested, *e.g.* $(A^{+} \wedge B^{+})^{-}$ is not a well-formed proposition.

The *opposite proposition* $P^{\sim}$ of a given proposition P is defined by flipping the sign, but preserving the strength:

$$\begin{array}{rclcrcl} (A^{+})^{\sim} & \stackrel{\text{def}}{=} & A^{-} & \quad & (A^{-})^{\sim} & \stackrel{\text{def}}{=} & A^{+} \\ (A^{\oplus})^{\sim} & \stackrel{\text{def}}{=} & A^{\ominus} & \quad & (A^{\ominus})^{\sim} & \stackrel{\text{def}}{=} & A^{\oplus} \end{array}$$

The *classical projection* of a given proposition P is written $\bigcirc P$ and is defined by preserving the sign, but losing its strength:

$$\begin{array}{rclcrcl} \bigcirc(A^{+}) & \stackrel{\text{def}}{=} & A^{\oplus} & \quad & \bigcirc(A^{-}) & \stackrel{\text{def}}{=} & A^{\ominus} \\ \bigcirc(A^{\oplus}) & \stackrel{\text{def}}{=} & A^{\oplus} & \quad & \bigcirc(A^{\ominus}) & \stackrel{\text{def}}{=} & A^{\ominus} \end{array}$$

Note that $A^{\oplus}$ and $A^{\ominus}$ are fixed points of $\bigcirc$. Moreover, note that $P^{\sim\sim} = P$, $\bigcirc\bigcirc P = \bigcirc P$, and $\bigcirc(P^{\sim}) = (\bigcirc P)^{\sim}$. That is, $^{\sim}$ is involutive, $\bigcirc$ is idempotent, and these operators commute with each other.

Definition 1 (System PRK). Judgments in PRK are of the form $\Gamma \vdash P$, where Γ is a finite *set* of moded propositions, *i.e.* we work implicitly up to structural rules of contraction and exchange. Derivability of judgments is defined inductively by the following inference schemes.

$$\frac{}{\Gamma, P \vdash P}\ \textsc{Ax} \qquad \frac{\Gamma \vdash A^+ \quad \Gamma \vdash A^-}{\Gamma \vdash P}\ \textsc{Abs}$$

$$\frac{\Gamma \vdash A^\oplus \quad \Gamma \vdash B^\oplus}{\Gamma \vdash (A \wedge B)^+}\ \mathrm{I}\wedge^+ \qquad \frac{\Gamma \vdash A^\ominus \quad \Gamma \vdash B^\ominus}{\Gamma \vdash (A \vee B)^-}\ \mathrm{I}\vee^-$$

$$\frac{\Gamma \vdash (A_1 \wedge A_2)^+ \quad i \in \{1,2\}}{\Gamma \vdash A_i{}^\oplus}\ \mathrm{E}\wedge_i^+$$

$$\frac{\Gamma \vdash (A_1 \vee A_2)^- \quad i \in \{1,2\}}{\Gamma \vdash A_i{}^\ominus}\ \mathrm{E}\vee_i^-$$

$$\frac{\Gamma \vdash A_i{}^\oplus \quad i \in \{1,2\}}{\Gamma \vdash (A_1 \vee A_2)^+}\ \mathrm{I}\vee_i^+ \qquad \frac{\Gamma \vdash A_i{}^\ominus \quad i \in \{1,2\}}{\Gamma \vdash (A_1 \wedge A_2)^-}\ \mathrm{I}\wedge_i^-$$

$$\frac{\Gamma \vdash (A \vee B)^+ \quad \Gamma, A^\oplus \vdash P \quad \Gamma, B^\oplus \vdash P}{\Gamma \vdash P}\ \mathrm{E}\vee^+$$

$$\frac{\Gamma \vdash (A \wedge B)^- \quad \Gamma, A^\ominus \vdash P \quad \Gamma, B^\ominus \vdash P}{\Gamma \vdash P}\ \mathrm{E}\wedge^-$$

$$\frac{\Gamma \vdash A^\ominus}{\Gamma \vdash (\neg A)^+}\ \mathrm{I}\neg^+ \qquad \frac{\Gamma \vdash A^\oplus}{\Gamma \vdash (\neg A)^-}\ \mathrm{I}\neg^-$$

$$\frac{\Gamma \vdash (\neg A)^+}{\Gamma \vdash A^\ominus}\ \mathrm{E}\neg^+ \qquad \frac{\Gamma \vdash (\neg A)^-}{\Gamma \vdash A^\oplus}\ \mathrm{E}\neg^-$$

$$\frac{\Gamma, A^\ominus \vdash A^+}{\Gamma \vdash A^\oplus}\ \mathrm{IC}^+ \qquad \frac{\Gamma, A^\oplus \vdash A^-}{\Gamma \vdash A^\ominus}\ \mathrm{IC}^-$$

$$\frac{\Gamma \vdash A^\oplus \quad \Gamma \vdash A^\ominus}{\Gamma \vdash A^+}\ \mathrm{EC}^+ \qquad \frac{\Gamma \vdash A^\ominus \quad \Gamma \vdash A^\oplus}{\Gamma \vdash A^-}\ \mathrm{EC}^-$$

Rule Ax is the standard axiom rule. Rule Abs expresses an explosion principle, allowing one to conclude any proposition from two opposite strong propositions. Rules $\mathrm{I}\wedge^+$ and $\mathrm{I}\vee^-$ introduce, respectively, a strong positive conjunction and a strong negative disjunction, by means of combining classical positive and negative propositions. This symmetry between dual operators with opposite sign will continue to show up in the incoming rules, and the rest of the thesis. Rules $\mathrm{E}\wedge_i^+$ and $\mathrm{E}\vee_i^-$ are the eliminators for the positive conjunction and the negative disjunction.

Rules $\mathrm{I}\vee_i^+$ and $\mathrm{I}\wedge_i^-$ introduce strong positive disjunction and strong negative conjunction, from a classical proof of one of its parts. And $\mathrm{E}\vee^+$ and $\mathrm{E}\wedge^-$ eliminate those moded connectives. Rules $\mathrm{I}\neg^+$ and $\mathrm{I}\neg^-$ introduce negation, positively and negatively; while $\mathrm{E}\neg^+$

and E$\neg^-$ eliminate negation. Note that, negation being its own dual, the symmetry appears with itself.

Introduction and elimination of classical modes (both negative and positive) are taken care by rules IC$^+$ and IC$^-$, for classical introduction, and rules EC$^+$ and EC$^-$, for classical elimination.

Finally notice that introduction rules for strong propositions use classical propositions to derive strong ones, whereas elimination rules for strong propositions use strong propositions to derive classical ones.

2 Admissible rules and basic properties

Before moving forward with the study of PRK, we present some basic properties and admissible rules that will be used freely, sometimes without explicit mention, throughout the thesis.

Lemma 2. *The following inference schemes are admissible in* PRK*:*

- **Weakening** *(*W*): if* $\Gamma \vdash P$ *then* $\Gamma, Q \vdash P$*.*
- **Cut** *(*Cut*): if* $\Gamma, P \vdash Q$ *and* $\Gamma \vdash P$ *then* $\Gamma \vdash Q$*.*
- **Substitution** *(*Sub*): if* $\Gamma \vdash Q$ *then* $\Gamma[\alpha := A] \vdash Q[\alpha := A]$*, where* $-[\alpha := A]$ *denotes the substitution of the propositional variable* α *for the pure proposition* A*.*
- **General absurdity** *(*Abs$'$*): if* $\Gamma \vdash P$ *and* $\Gamma \vdash P^\sim$*, where* P *is not necessarily strong, then* $\Gamma \vdash Q$*.*
- **Projection of conclusions** *(*PC*): if* $\Gamma \vdash P$ *then* $\Gamma \vdash \bigcirc P$*.*
- **Injection of premises** *(*IP*): if* $\Gamma, \bigcirc Q \vdash P$ *then* $\Gamma, Q \vdash P$*.*
- **Contraposition** *(*Contra*): if* P *is classical and* $\Gamma, P \vdash Q$ *then* $\Gamma, Q^\sim \vdash P^\sim$*.*
- **Classical strengthening** *(*CS*): if* P *is classical and* $\Gamma, P^\sim \vdash P$ *then* $\Gamma \vdash P$*.*

Proof. **Weakening**, **cut**, and **substitution** are routine proofs by induction on the derivation of the first judgment.

For **general absurdity**, suppose that $\Gamma \vdash P$ and $\Gamma \vdash P^\sim$. If P is strong and positive, applying the Abs rule we may conclude $\Gamma \vdash Q$. If P is strong and negative, applying the Abs rule on $P^\sim$ and P is enough. If P is classical, there are two cases, depending on whether P is positive or negative. If P is positive, *i.e.* $P = A^\oplus$ then:

$$\dfrac{\dfrac{\Gamma \vdash P \quad \Gamma \vdash P^\sim}{\Gamma \vdash A^+}\text{EC}^+ \quad \dfrac{\Gamma \vdash P^\sim \quad \Gamma \vdash P}{\Gamma \vdash A^-}\text{EC}^-}{\Gamma \vdash Q}\text{Abs}$$

If P is negative, *i.e.* $P = A^\ominus$, the proof is symmetric.

For **projection of conclusions**, if P is classical, *i.e.* of the form $A^{\oplus}$ or $A^{\ominus}$, we are done. If P is strong, *i.e.* of the form A^{+} or A^{-}, we conclude by applying the IC^{+} or the IC^{-} rule respectively, followed by a Weakening. For example, if $P = A^{+}$:

$$\dfrac{\dfrac{\Gamma \vdash A^{+}}{\Gamma, A^{\ominus} \vdash A^{+}}\,\mathrm{W}}{\Gamma \vdash A^{\oplus}}\,\mathrm{IC}^{+}$$

For **injection of premises**, we proceed by induction on the derivation of the first judgment. The interesting case is when using the $\textsc{Ax}$ rule over $\bigcirc Q$; if Q is classical, the same $\textsc{Ax}$ rule can be used. Otherwise, $\textsc{Ax}$ followed by PC gives us what we need.

For **contraposition** we only study the case when P is positive, *i.e.* $P = A^{\oplus}$; the negative case is symmetric. So let $\Gamma, A^{\oplus} \vdash Q$. Then:

$$\dfrac{\dfrac{\dfrac{\Gamma, A^{\oplus} \vdash Q}{\Gamma, Q^{\sim}, A^{\oplus} \vdash Q}\,\mathrm{W} \qquad \dfrac{}{\Gamma, Q^{\sim}, A^{\oplus} \vdash Q^{\sim}}\,\textsc{Ax}}{\Gamma, Q^{\sim}, A^{\oplus} \vdash A^{-}}\,\textsc{Abs}'}{\Gamma, Q^{\sim} \vdash A^{\ominus}}\,\mathrm{IC}^{-}$$

For **classical strengthening** we only study the case when P is positive, *i.e.* $P = A^{\oplus}$; the negative case is symmetric. So let $\Gamma, A^{\ominus} \vdash A^{\oplus}$. Then:

$$\dfrac{\dfrac{\Gamma, A^{\ominus} \vdash A^{\oplus} \qquad \dfrac{}{\Gamma, A^{\ominus} \vdash A^{\ominus}}\,\textsc{Ax}}{\Gamma, A^{\ominus} \vdash A^{+}}\,\mathrm{EC}^{+}}{\Gamma \vdash A^{\oplus}}\,\mathrm{IC}^{+}$$

□

3 Examples and properties

Projection Lemma. The proof of the following lemma is subtle. It will be a key tool in order to prove completeness of PRK with respect to the Kripke semantics in the next chapter:

Lemma 3 (Projection)**.** *If* $\Gamma, P \vdash Q$ *then* $\Gamma, \bigcirc P \vdash \bigcirc Q$*.*

Proof. We call P the *target assumption.* The proof proceeds by induction on the derivation of $\Gamma, P \vdash Q$. We only study the cases with positive signs, the negative cases are symmetric.

- **Ax**: let $\Gamma, Q \vdash Q$. There are two cases, depending on whether the target assumption is in Γ or if it is Q.
    1. *If the target assumption is in* Γ*,* i.e. $\Gamma = \Gamma', P$. Note that we have $\Gamma', \bigcirc P, Q \vdash Q$ by the $\textsc{Ax}$ rule. By projecting the conclusion (Lem. 2) we conclude that $\Gamma', \bigcirc P, Q \vdash \bigcirc Q$, as required.
    2. *If the target assumption is* Q*.* Then we have that $\Gamma, \bigcirc Q \vdash \bigcirc Q$ by the $\textsc{Ax}$ rule.
- **Abs**: let $\Gamma, P \vdash Q$ be derived from $\Gamma, P \vdash A^{+}$ and $\Gamma, P \vdash A^{-}$ for some pure proposition A. By IH we have that $\Gamma, \bigcirc P \vdash A^{\oplus}$ and $\Gamma, \bigcirc P \vdash A^{\ominus}$ so by the generalized absurdity rule ($\textsc{Abs}'$) we have that $\Gamma, \bigcirc P \vdash \bigcirc Q$.

- **I**$\wedge^+$: let $\Gamma, P \vdash (A \wedge B)^+$ be derived from $\Gamma, P \vdash A^\oplus$ and $\Gamma, P \vdash B^\oplus$. By IH, $\Gamma, \bigcirc P \vdash A^\oplus$ and $\Gamma, \bigcirc P \vdash B^\oplus$. By the I$\wedge^+$ rule, $\Gamma, \bigcirc P \vdash (A \wedge B)^+$. Projecting the conclusion (Lem. 2), we obtain $\Gamma, \bigcirc P \vdash (A \wedge B)^\oplus$ as required.

- **E**$\wedge_i^+$: let $\Gamma, P \vdash A_i{}^\oplus$ be derived from $\Gamma, P \vdash (A_1 \wedge A_2)^+$. Then the proof is of the form:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P \vdash (A_1 \wedge A_2)^\oplus}}{\Gamma, \bigcirc P, A_i{}^\ominus \vdash (A_1 \wedge A_2)^\oplus}\,\text{W} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \bigcirc P, A_i{}^\ominus \vdash A_i{}^\ominus}\,\text{Ax}}{\Gamma, \bigcirc P, A_i{}^\ominus \vdash (A_1 \wedge A_2)^-}\,\text{I}\wedge_i^-}{\Gamma, \bigcirc P, A_i{}^\ominus \vdash (A_1 \wedge A_2)^\ominus}\,\text{PC}}{\dfrac{\Gamma, \bigcirc P, A_i{}^\ominus \vdash (A_1 \wedge A_2)^+}{\dfrac{\Gamma, \bigcirc P, A_i{}^\ominus \vdash A_i{}^\oplus}{\Gamma, \bigcirc P \vdash A_i{}^\oplus}\,\text{CS}}\,\text{E}\wedge_i^+}\,\text{EC}^+
$$

- **I**$\vee_i^+$: let $\Gamma, P \vdash (A_1 \vee A_2)^+$ be derived from $\Gamma, P \vdash A_i{}^\oplus$. By IH, $\Gamma, \bigcirc P \vdash A_i{}^\oplus$. By the I$\vee_i^+$ rule, $\Gamma, \bigcirc P \vdash (A_1 \vee A_2)^+$. Projecting the conclusion (Lem. 2), we reach $\Gamma, \bigcirc P \vdash (A_1 \vee A_2)^\oplus$.

- **E**$\vee^+$: let $\Gamma, P \vdash Q$ be derived from $\Gamma, P \vdash (A_1 \vee A_2)^+$ and $\Gamma, P, A_i{}^\oplus \vdash Q$ for each $i \in \{1, 2\}$. By IH, $\Gamma, \bigcirc P \vdash (A_1 \vee A_2)^\oplus$ and $\Gamma, \bigcirc P, A_i{}^\oplus \vdash \bigcirc Q$ for each $i \in \{1, 2\}$. Then the proof is of the form:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P \vdash (A_1 \vee A_2)^\oplus}}{\Gamma, \bigcirc P, \bigcirc Q^\sim \vdash (A_1 \vee A_2)^\oplus}\,\text{W} \qquad \dfrac{\begin{matrix}\vdots & \vdots \\ \xi_1 & \xi_2\end{matrix}}{\Gamma, \bigcirc P, \bigcirc Q^\sim \vdash (A_1 \vee A_2)^\ominus}\,\text{I}\vee^-}{\Gamma, \bigcirc P, \bigcirc Q^\sim \vdash (A_1 \vee A_2)^+}\,\text{EC}^+ \quad \begin{matrix}\vdots \\ \pi_1\end{matrix} \quad \begin{matrix}\vdots \\ \pi_2\end{matrix}
$$

$$
\dfrac{\dfrac{\cdots}{\Gamma, \bigcirc P, \bigcirc Q^\sim \vdash \bigcirc Q}\,\text{E}\vee^+}{\Gamma, \bigcirc P \vdash \bigcirc Q}\,\text{CS}
$$

where for each $i \in \{1, 2\}$ the derivations π_i and ξ_i are given by:

$$
\pi_i \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P, A_i{}^\oplus \vdash \bigcirc Q}}{\Gamma, \bigcirc P, \bigcirc Q^\sim, A_i{}^\oplus \vdash \bigcirc Q}\,\text{W} \right)
$$

$$
\xi_i \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P, A_i{}^\oplus \vdash \bigcirc Q}}{\Gamma, \bigcirc P, \bigcirc Q^\sim \vdash A_i{}^\ominus}\,\textsc{Contra} \right)
$$

- **I**$\neg^+$: let $\Gamma, P \vdash (\neg A)^+$ be derived from $\Gamma, P \vdash A^\ominus$. By IH we have that $\Gamma, \bigcirc P \vdash A^\ominus$. By the I$\neg^+$ rule, $\Gamma, \bigcirc P \vdash (\neg A)^+$. Projecting the conclusion (Lem. 2), we obtain $\Gamma, \bigcirc P \vdash (\neg A)^\oplus$.

- **E$\neg^+$**: let $\Gamma, P \vdash A^\ominus$ be derived from $\Gamma, P \vdash (\neg A)^+$. Then the proof is of the form:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P \vdash (\neg A)^\oplus}}{\Gamma, \bigcirc P, A^\oplus \vdash (\neg A)^\oplus}\,\text{W}
      \qquad
      \dfrac{\dfrac{\dfrac{}{\Gamma, \bigcirc P, A^\oplus, (\neg A)^\oplus \vdash A^\oplus}\,\text{Ax}}{\Gamma, \bigcirc P, A^\oplus, (\neg A)^\oplus \vdash (\neg A)^-}\,\text{I}\neg^-}{\Gamma, \bigcirc P, A^\oplus \vdash (\neg A)^\ominus}\,\text{IC}^-
    }{\Gamma, \bigcirc P, A^\oplus \vdash (\neg A)^+}\,\text{EC}^+
  }{\Gamma, \bigcirc P, A^\oplus \vdash A^\ominus}\,\text{E}\neg^+
}{\Gamma, \bigcirc P \vdash A^\ominus}\,\text{CS}
$$

- **IC$^+$**: let $\Gamma, P \vdash A^\oplus$ be derived from $\Gamma, P, A^\ominus \vdash A^+$. By IH, $\Gamma, \bigcirc P, A^\ominus \vdash A^\oplus$, so by classical strengthening (Lem. 2) we have that $\Gamma, \bigcirc P \vdash A^\oplus$.

- **EC$^+$**: let $\Gamma, P \vdash A^+$ be derived from $\Gamma, P \vdash A^\oplus$ and $\Gamma, P \vdash A^\ominus$. Then, in particular, by IH on the first premise, we have $\Gamma, \bigcirc P \vdash A^\oplus$, as required.

□

A corollary obtained from iterating the projection lemma and considering the idempotence of $\bigcirc$ is that if $P_1, \ldots, P_n \vdash Q$ then $\bigcirc P_1, \ldots, \bigcirc P_n \vdash \bigcirc Q$.

Duality Principle. The *dual* of a pure proposition A is written $A^\perp$ and defined as:

$$
\begin{array}{rclcrcl}
\alpha^\perp & \stackrel{\text{def}}{=} & \alpha & \quad & (A \wedge B)^\perp & \stackrel{\text{def}}{=} & A^\perp \vee B^\perp \\
(A \vee B)^\perp & \stackrel{\text{def}}{=} & A^\perp \wedge B^\perp & & (\neg A)^\perp & \stackrel{\text{def}}{=} & \neg(A^\perp)
\end{array}
$$

The dual of a proposition P is written $P^\perp$ and defined as:

$$
\begin{array}{rclcrcl}
(A^+)^\perp & \stackrel{\text{def}}{=} & (A^\perp)^- & \quad & (A^-)^\perp & \stackrel{\text{def}}{=} & (A^\perp)^+ \\
(A^\oplus)^\perp & \stackrel{\text{def}}{=} & (A^\perp)^\ominus & & (A^\ominus)^\perp & \stackrel{\text{def}}{=} & (A^\perp)^\oplus
\end{array}
$$

The following duality principle is then straightforward to prove by induction on the derivation of the judgment:

Lemma 4. *If $P_1, \ldots, P_n \vdash Q$ then $P_1^\perp, \ldots, P_n^\perp \vdash Q^\perp$.*

Example 5 (Law of excluded middle)**.** *The law of excluded middle holds classically in* PRK*, that is, $\vdash (A \vee \neg A)^\oplus$. Indeed, let $\Gamma = \{(A \vee \neg A)^\ominus, (\neg A)^\ominus\}$, and let π be the following* *derivation:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A^{\ominus} \vdash (\neg A)^{\ominus}}\,\text{Ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A^{\ominus} \vdash A^{\ominus}}\,\text{Ax}}{\Gamma, A^{\ominus} \vdash (\neg A)^{+}}\,\text{I}\neg^{+}}{(A \vee \neg A)^{\ominus}, A^{\ominus} \vdash (\neg A)^{\oplus}}\,\text{IC}^{+}}{\Gamma, A^{\ominus} \vdash (\neg A)^{\oplus}}\,\text{W}}{\Gamma, A^{\ominus} \vdash A^{+}}\,\text{Abs}'}{\dfrac{\Gamma \vdash A^{\oplus}}{\dfrac{\Gamma \vdash (A \vee \neg A)^{+}}{(\neg A)^{\ominus} \vdash (A \vee \neg A)^{\oplus}}\,\text{IC}^{+}}\,\text{I}\vee_1^{+}}\,\text{IC}^{+}}{\Gamma \vdash (A \vee \neg A)^{\oplus}}\,\text{W}
$$

Then we have that:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus} \vdash (A \vee \neg A)^{\ominus}}\,\text{Ax} \quad \begin{matrix}\vdots \\ \pi\end{matrix}}{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus} \vdash (A \vee \neg A)^{-}}\,\text{EC}^{-}}{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus} \vdash A^{\ominus}}\,\text{E}\vee_1^{-}}{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus} \vdash (\neg A)^{+}}\,\text{I}\neg^{+}}{\dfrac{(A \vee \neg A)^{\ominus} \vdash (\neg A)^{\oplus}}{(A \vee \neg A)^{\ominus} \vdash (A \vee \neg A)^{+}}\,\text{I}\vee_2^{+}}\,\text{IC}^{+}}{\vdash (A \vee \neg A)^{\oplus}}\,\text{IC}^{+}
$$

From the duality principle, the law of non-contradiction holds classically in PRK*, that is,* $\vdash (A \wedge \neg A)^{\ominus}$ *holds.*

Results from the following chapter will entail that the strong law of excluded middle, $\vdash (A \vee \neg A)^{+}$*, does not hold in* PRK *(see Ex. 24). The reader may attempt to derive this judgment to convince herself that it does not hold.*

Remark 6 (Dispensable eliminators). A property of the PRK system that may seem striking is that some of the eliminators presented above for strong propositions, that is $\text{E}\wedge_i^{+}$, $\text{E}\vee_i^{-}$, $\text{E}\neg^{+}$, and $\text{E}\neg^{-}$, are actually not needed from the strictly *logical point of view*, as they can be derived using the other rules, and in particular the Abs rule.

As an example, this is how we could express the $\text{E}\wedge_1^{+}$ rule:

$$
\dfrac{\dfrac{\dfrac{\Gamma \vdash (A \wedge B)^{+}}{\Gamma, A^{\ominus} \vdash (A \wedge B)^{+}}\,\text{W} \quad \dfrac{\dfrac{}{\Gamma, A^{\ominus} \vdash A^{\ominus}}\,\text{Ax}}{\Gamma, A^{\ominus} \vdash (A \wedge B)^{-}}\,\text{I}\wedge_1^{-}}{\Gamma, A^{\ominus} \vdash A^{+}}\,\text{Abs}}{\Gamma \vdash A^{\oplus}}\,\text{IC}^{+}
$$

In a similar way, the elimination rules $\text{E}\wedge^{-}$ and $\text{E}\vee^{+}$ can be derived using Abs, as long as the proposition P is restricted to be classical. Even if this observation would allow us to simplify the logical system by reducing the number of inference schemes, once the *computational point of view* comes into play in the next chapters, the behaviour of the elimination rules (seen as axioms) will not coincide with the behaviour of their indirect proofs. This is why we postulate them as separate rules from the start.

3. KRIPKE SEMANTICS

In this chapter we will define a semantics for the PRK logical system based on Kripke models, and prove its soundness and completeness. Also, we will show the validity of some of the examples presented earlier using Kripke models, as well as some extra expected results.

1 Definition of Kripke model

Recall that in intuitionistic logic[1] a Kripke model $\mathcal{M}$ is given by a set $\mathcal{W}$ of elements called *worlds*, a partial order $\leq$ on $\mathcal{W}$ called the *accessibility relation*, and for each world $w \in \mathcal{W}$ a set $\mathcal{V}_w$ of propositional variables verifying a *monotonicity* property, namely, that $w \leq w'$ implies $\mathcal{V}_w \subseteq \mathcal{V}_{w'}$. A relation of forcing $\mathcal{M}, w \Vdash A$ is defined by structural recursion on A. In the base case, $\mathcal{M}, w \Vdash \alpha$ is declared to hold for a propositional variable α whenever $\alpha \in \mathcal{V}_w$.

This standard notion of Kripke model is adapted for PRK by replacing the set $\mathcal{V}_w$ with two sets $\mathcal{V}_w^+$ and $\mathcal{V}_w^-$, a positive and a negative one; and by imposing an additional condition we name *stabilization*, stating that a propositional variable must eventually belong to the union $\mathcal{V}_w^+ \cup \mathcal{V}_w^-$, but never to the intersection $\mathcal{V}_w^+ \cap \mathcal{V}_w^-$. The relation of forcing $\mathcal{M}, w \Vdash P$ is then defined in such a way that $\mathcal{M}, w \Vdash \alpha^+$ is declared to hold if $\alpha \in \mathcal{V}_w^+$, and α^- is declared to hold if $\alpha \in \mathcal{V}_w^-$.

One difficulty that we found is how to define the forcing relation for a classical proposition like $A^\oplus$. The forcing relation for $A^\oplus$ should behave, informally speaking, like an intuitionistic implication "$A^\ominus \to A^+$". However this does not provide a *bona fide* definition, because the interpretation of $A^\oplus$ would depend on $A^\ominus$, and the interpretation of $A^\ominus$ would, in turn, depend on $A^\oplus$. What we do is define the interpretations of $A^\oplus$ and $A^\ominus$ without referring to each other. A key lemma (Lem. 12) then ensures that $A^\oplus$ is given the same semantics as an intuitionistic implication of the form "$A^\ominus \to A^+$".

Definition 7. A *Kripke model* (for PRK) is a structure $\mathcal{M} = (\mathcal{W}, \leq, \mathcal{V}^+, \mathcal{V}^-)$ where $\mathcal{W} = \{w, w', \ldots\}$ is a set of worlds, $\leq$ is a partial order on $\mathcal{W}$, and for each world $w \in \mathcal{W}$ there are sets $\mathcal{V}_w^+$ and $\mathcal{V}_w^-$ of propositional variables, such that the following conditions hold:

1. **Monotonicity.** If $w \leq w'$ then $\mathcal{V}_w^+ \subseteq \mathcal{V}_{w'}^+$ and $\mathcal{V}_w^- \subseteq \mathcal{V}_{w'}^-$.
2. **Stabilization.** For all $w \in \mathcal{W}$ and all α, there exists $w' \geq w$ such that $\alpha \in \mathcal{V}_{w'}^+ \triangle \mathcal{V}_{w'}^-$.

Note that we write $w' \geq w$ for $w \leq w'$, and $\triangle$ denotes the symmetric difference on sets, that is, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.

It's worth noting that this definition of Kripke model has various similarities to the one presented by Ilik, Lee, and Herbelin [9] for classical logic, but differs on some key aspects.

[1] See for instance [8, Section 5.3].

The definition of the forcing relation is given by induction on the following *measure* $\#(P)$ of a proposition P:

$$\begin{array}{rclrcl} \#(A^+) & \overset{\text{def}}{=} & 2|A| & \#(A^-) & \overset{\text{def}}{=} & 2|A| \\ \#(A^\oplus) & \overset{\text{def}}{=} & 2|A|+1 & \#(A^\ominus) & \overset{\text{def}}{=} & 2|A|+1 \end{array}$$

where $|A|$ denotes the *size*, *i.e.* the number of symbols, in the formula A. Note in particular that $\#(A^\oplus) = \#(A^\ominus) > \#(A^+) = \#(A^-)$, that $\#((A_1 \star A_2)^+) = \#((A_1 \star A_2)^-) = \#(A_1{}^\oplus) + \#(A_2{}^\oplus) = \#(A_1{}^\ominus) + \#(A_2{}^\ominus) > \#(A_i{}^\oplus) = \#(A_i{}^\ominus)$ for $\star \in \{\land, \lor\}$, and that $\#((\neg A)^+) = \#((\neg A)^-) > \#(A^\oplus) = \#(A^\ominus)$.

Definition 8 (Forcing)**.** Given a Kripke model, we define the *forcing* relation, written $\mathcal{M}, w \Vdash P$ for each world $w \in \mathcal{W}$ and each proposition P, as follows, by induction on the *measure* $\#(P)$:

$$\begin{array}{lcl} \mathcal{M}, w \Vdash \alpha^+ & \iff & \alpha \in \mathcal{V}_w^+ \\ \mathcal{M}, w \Vdash \alpha^- & \iff & \alpha \in \mathcal{V}_w^- \\ \mathcal{M}, w \Vdash (A \land B)^+ & \iff & \mathcal{M}, w \Vdash A^\oplus \ \text{ and } \ \mathcal{M}, w \Vdash B^\oplus \\ \mathcal{M}, w \Vdash (A \land B)^- & \iff & \mathcal{M}, w \Vdash A^\ominus \ \text{ or } \ \mathcal{M}, w \Vdash B^\ominus \\ \mathcal{M}, w \Vdash (A \lor B)^+ & \iff & \mathcal{M}, w \Vdash A^\oplus \ \text{ or } \ \mathcal{M}, w \Vdash B^\oplus \\ \mathcal{M}, w \Vdash (A \lor B)^- & \iff & \mathcal{M}, w \Vdash A^\ominus \ \text{ and } \ \mathcal{M}, w \Vdash B^\ominus \\ \mathcal{M}, w \Vdash (\neg A)^+ & \iff & \mathcal{M}, w \Vdash A^\ominus \\ \mathcal{M}, w \Vdash (\neg A)^- & \iff & \mathcal{M}, w \Vdash A^\oplus \\ \mathcal{M}, w \Vdash A^\oplus & \iff & \mathcal{M}, w' \nVdash A^- \text{ for all } w' \geq w \\ \mathcal{M}, w \Vdash A^\ominus & \iff & \mathcal{M}, w' \nVdash A^+ \text{ for all } w' \geq w \end{array}$$

Furthermore, if Γ is a (possibly infinite) set of propositions, we write:

$$\begin{array}{lcl} \mathcal{M}, w \Vvdash \Gamma & \iff & \mathcal{M}, w \Vdash P \text{ for every } P \in \Gamma \\ \mathcal{M}, \Gamma \Vvdash P & \iff & \mathcal{M}, w \Vvdash \Gamma \text{ implies } \mathcal{M}, w \Vdash P \text{ for every } w \\ \Gamma \Vvdash P & \iff & \mathcal{M}, \Gamma \Vvdash P \text{ for every Kripke model } \mathcal{M} \end{array}$$

Note that most cases in the definition of forcing do not mention the accessibility relation, other than for classical propositions.

Before moving forward, we introduce typical nomenclature. If Γ is a possibly infinite set of propositions, we say that $\Gamma \vdash P$ holds whenever the judgment $\Delta \vdash P$ is derivable in PRK for some finite subset $\Delta \subseteq \Gamma$. A set Γ of propositions is *consistent* if there is a proposition P such that $\Gamma \nvdash P$. Otherwise, Γ is *inconsistent.*

2 Forced properties and soundness

In this section and the next one we prove that PRK is sound and complete with respect to this notion of Kripke model. *i.e.* that $\Gamma \vdash P$ holds if and only if $\Gamma \Vvdash P$ holds. We begin by establishing some basic properties of the forcing relation.

Lemma 9 (Monotonicity of forcing)**.** *If $\mathcal{M}, w \Vdash P$ and $w \leq w'$ then $\mathcal{M}, w' \Vdash P$.*

Proof. By induction on the measure $\#(P)$. We only check the positive propositions; the negative cases are dual—*e.g.* the proof for $(A \land B)^-$ is symmetric to the proof for $(A \lor B)^+$:

- **Propositional variable** ($P = \alpha^+$): Let $\mathcal{M}, w \Vdash \alpha^+$, that is $\alpha \in \mathcal{V}_w^+$. Then by the monotonicity property we have that $\alpha \in \mathcal{V}_{w'}^+$, so $\mathcal{M}, w' \Vdash \alpha^+$.
- **Conjunction** ($P = (A \wedge B)^+$): Let $\mathcal{M}, w \Vdash (A \wedge B)^+$, that is $\mathcal{M}, w \Vdash A^\oplus$ and $\mathcal{M}, w \Vdash B^\oplus$. Then by IH $\mathcal{M}, w' \Vdash A^\oplus$ and $\mathcal{M}, w' \Vdash B^\oplus$ so $\mathcal{M}, w' \Vdash (A \wedge B)^+$.
- **Disjunction** ($P = (A_1 \vee A_2)^+$): Let $\mathcal{M}, w \Vdash (A_1 \vee A_2)^+$, that is $\mathcal{M}, w \Vdash A_i{}^\oplus$ for some $i \in \{1, 2\}$. Then by IH $\mathcal{M}, w' \Vdash A_i{}^\oplus$ so $\mathcal{M}, w' \Vdash (A_1 \vee A_2)^+$.
- **Negation** ($P = (\neg A)^+$): Let $\mathcal{M}, w \Vdash (\neg A)^+$, that is $\mathcal{M}, w \Vdash A^\ominus$. Then by IH $\mathcal{M}, w' \Vdash A^\ominus$ so $\mathcal{M}, w' \Vdash (\neg A)^+$.
- **Classical proposition** ($P = A^\oplus$): Let $\mathcal{M}, w \Vdash A^\oplus$, that is, for every $w'' \geq w$ we have that $\mathcal{M}, w'' \nVdash A^-$. Our goal is to prove that $\mathcal{M}, w' \Vdash A^\oplus$, so let $w'' \geq w'$ and let us check that $\mathcal{M}, w'' \nVdash A^-$. Indeed, given that $w'' \geq w' \geq w$ we have that $\mathcal{M}, w'' \nVdash A^-$.

□

Lemma 10 (Stabilization of forcing). *For every world w and every proposition P, there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash P$ or $\mathcal{M}, w' \Vdash P^\sim$, but not both.*

Proof. By induction on the measure $\#(P)$. We only check the positive propositions; the negative cases are symmetric.

- **Propositional variable** ($P = \alpha^+$ and $P^\sim = \alpha^-$): By the stabilization property, there exists $w' \geq w$ such that $\alpha \in \mathcal{V}_{w'}^+ \triangle \mathcal{V}_{w'}^-$, *i.e.* $\alpha \in \mathcal{V}_{w'}^+$ or $\alpha \in \mathcal{V}_{w'}^-$ but not both, so we consider two cases:
  1. If $\alpha \in \mathcal{V}_{w'}^+ \setminus \mathcal{V}_{w'}^-$ then $\mathcal{M}, w' \Vdash \alpha^+$ and $\mathcal{M}, w' \nVdash \alpha^-$.
  2. If $\alpha \in \mathcal{V}_{w'}^- \setminus \mathcal{V}_{w'}^+$ then $\mathcal{M}, w' \Vdash \alpha^-$ and $\mathcal{M}, w' \nVdash \alpha^+$.
- **Conjunction** ($P = (A \wedge B)^+$ and $P^\sim = (A \wedge B)^-$): By IH there is a world $w_1 \geq w$ such that either $\mathcal{M}, w_1 \Vdash A^\oplus$ or $\mathcal{M}, w_1 \Vdash A^\ominus$ but not both, so we consider two subcases:
  1. If $\mathcal{M}, w_1 \Vdash A^\oplus$ and $\mathcal{M}, w_1 \nVdash A^\ominus$, then by IH there is a world $w_2 \geq w_1$ such that either $\mathcal{M}, w_2 \Vdash B^\oplus$ or $\mathcal{M}, w_2 \Vdash B^\ominus$ but not both, so we consider two further subcases:
     1.1 If $\mathcal{M}, w_2 \Vdash B^\oplus$ and $\mathcal{M}, w_2 \nVdash B^\ominus$, then we take $w' := w_2$. By monotonicity (Lem. 9) we have that $\mathcal{M}, w_2 \Vdash A^\oplus$ so indeed $\mathcal{M}, w_2 \Vdash (A \wedge B)^+$. We are left to show that $\mathcal{M}, w_2 \nVdash (A \wedge B)^-$. We already know that $\mathcal{M}, w_2 \nVdash B^\ominus$, so to conclude it suffices to show that $\mathcal{M}, w_2 \nVdash A^\ominus$. Indeed, suppose that $\mathcal{M}, w_2 \Vdash A^\ominus$ holds. By IH there exists $w_3 \geq w_2$ such that either $\mathcal{M}, w_3 \Vdash A^\oplus$ or $\mathcal{M}, w_3 \Vdash A^\ominus$ but *not both*. However, by monotonicity (Lem. 9) —given that both $\mathcal{M}, w_2 \Vdash A^\oplus$ and $\mathcal{M}, w_2 \Vdash A^\ominus$ hold— we know that both $\mathcal{M}, w_3 \Vdash A^\oplus$ and $\mathcal{M}, w_3 \Vdash A^\ominus$ hold, a contradiction.

     1.2 If $\mathcal{M}, w_2 \Vdash B^\ominus$ and $\mathcal{M}, w_2 \nVdash B^\oplus$, then we take $w' := w_2$, and we have that $\mathcal{M}, w_2 \Vdash (A \wedge B)^-$ and $\mathcal{M}, w_2 \nVdash (A \wedge B)^+$.
  2. If $\mathcal{M}, w_1 \Vdash A^\ominus$ and $\mathcal{M}, w_1 \nVdash A^\oplus$, then we take $w' := w_1$, and we have that $\mathcal{M}, w_1 \Vdash (A \wedge B)^-$ and $\mathcal{M}, w_1 \nVdash (A \wedge B)^+$.

- **Disjunction** ($P = (A \vee B)^+$ and $P^\sim = (A \vee B)^-$): By IH there is a world $w_1 \geq w$ such that either $\mathcal{M}, w_1 \Vdash A^\oplus$ or $\mathcal{M}, w_1 \Vdash A^\ominus$ but not both, so we consider two subcases:

  1. If $\mathcal{M}, w_1 \Vdash A^\oplus$ and $\mathcal{M}, w_1 \nVdash A^\ominus$, then we take $w' := w_1$, and we have that $\mathcal{M}, w_1 \Vdash (A \vee B)^+$ and $\mathcal{M}, w_1 \nVdash (A \vee B)^-$.
  2. If $\mathcal{M}, w_1 \Vdash A^\ominus$ and $\mathcal{M}, w_1 \nVdash A^\oplus$, then by IH there is a world $w_2 \geq w_1$ such that either $\mathcal{M}, w_2 \Vdash B^\oplus$ or $\mathcal{M}, w_2 \Vdash B^\ominus$ but not both, so we consider two further subcases:

     2.1 If $\mathcal{M}, w_2 \Vdash B^\oplus$ and $\mathcal{M}, w_2 \nVdash B^\ominus$, then we take $w' := w_2$, and we have that $\mathcal{M}, w_2 \Vdash (A \vee B)^+$ and $\mathcal{M}, w_2 \nVdash (A \vee B)^-$.

     2.2 If $\mathcal{M}, w_2 \Vdash B^\ominus$ and $\mathcal{M}, w_2 \nVdash B^\oplus$, then we take $w' := w_2$. By monotonicity (Lem. 9) we have that $\mathcal{M}, w_2 \Vdash A^\ominus$ so indeed $\mathcal{M}, w_2 \Vdash (A \vee B)^-$. We are left to show that $\mathcal{M}, w_2 \nVdash (A \vee B)^+$. We already know that $\mathcal{M}, w_2 \nVdash B^\oplus$, so we are left to show that $\mathcal{M}, w_2 \nVdash A^\oplus$. Indeed, suppose that $\mathcal{M}, w_2 \Vdash A^\oplus$ holds. By IH there exists $w_3 \geq w_2$ such that either $\mathcal{M}, w_3 \Vdash A^\oplus$ or $\mathcal{M}, w_3 \Vdash A^\ominus$ holds but *not both*. However, by monotonicity (Lem. 9) —given that both $\mathcal{M}, w_2 \Vdash A^\oplus$ and $\mathcal{M}, w_2 \Vdash A^\ominus$ hold— we know that both $\mathcal{M}, w_3 \Vdash A^\oplus$ and $\mathcal{M}, w_3 \Vdash A^\ominus$ hold, a contradiction.

- **Negation** ($P = (\neg A)^+$ and $P^\sim = (\neg A)^-$): By IH there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash A^\oplus$ or $\mathcal{M}, w' \Vdash A^\ominus$ hold but not both, so we consider two cases:

  1. If $\mathcal{M}, w' \Vdash A^\oplus$ and $\mathcal{M}, w' \nVdash A^\ominus$, then $\mathcal{M}, w' \Vdash (\neg A)^-$ and $\mathcal{M}, w' \nVdash (\neg A)^+$.
  2. If $\mathcal{M}, w' \Vdash A^\ominus$ and $\mathcal{M}, w' \nVdash A^\oplus$, then $\mathcal{M}, w' \Vdash (\neg A)^+$ and $\mathcal{M}, w' \nVdash (\neg A)^-$.

- **Classical proposition** ($P = A^\oplus$ and $P^\sim = A^\ominus$): By IH there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash A^+$ or $\mathcal{M}, w' \Vdash A^-$ but not both. We consider two subcases:

  1. If $\mathcal{M}, w' \Vdash A^+$ and $\mathcal{M}, w' \nVdash A^-$, then we claim that $\mathcal{M}, w' \Vdash A^\oplus$ and $\mathcal{M}, w' \nVdash A^\ominus$. Indeed, let us prove each condition:

     1.1 In order to show that $\mathcal{M}, w' \Vdash A^\oplus$, it suffices to check that given $w'' \geq w'$ we have that $\mathcal{M}, w'' \nVdash A^-$. Indeed, suppose that $\mathcal{M}, w'' \Vdash A^-$. Then by IH there exists $w''' \geq w''$ such that either $\mathcal{M}, w''' \Vdash A^+$ or $\mathcal{M}, w''' \Vdash A^-$ but *not both*. However, by monotonicity (Lem. 9) —given that both $\mathcal{M}, w' \Vdash A^+$ and $\mathcal{M}, w'' \Vdash A^-$ hold, and $w' \leq w'' \leq w'''$— we know that both $\mathcal{M}, w''' \Vdash A^+$ and $\mathcal{M}, w''' \Vdash A^-$ hold, a contradiction.

     1.2 In order to show that $\mathcal{M}, w' \nVdash A^\ominus$, it suffices to note that $\mathcal{M}, w' \Vdash A^+$, which contradicts the definition of $\mathcal{M}, w' \Vdash A^\ominus$, given that accessibility is reflexive, *i.e.* $w' \leq w'$.
  2. If $\mathcal{M}, w' \Vdash A^-$ and $\mathcal{M}, w' \nVdash A^+$, then we claim that $\mathcal{M}, w' \Vdash A^\ominus$ and $\mathcal{M}, w' \nVdash A^\oplus$. Indeed, let us prove each condition:

     2.1 In order to show that $\mathcal{M}, w' \Vdash A^\ominus$, it suffices to check that given $w'' \geq w'$ we have that $\mathcal{M}, w'' \nVdash A^+$. Indeed, suppose that $\mathcal{M}, w'' \Vdash A^+$. Then by IH there exists $w''' \geq w''$ such that either $\mathcal{M}, w''' \Vdash A^+$ and $\mathcal{M}, w''' \Vdash A^-$ but *not both*. However, by monotonicity (Lem. 9) —given that both $\mathcal{M}, w'' \Vdash A^+$ and $\mathcal{M}, w' \Vdash A^-$ hold, and $w' \leq w'' \leq w'''$— we know that both $\mathcal{M}, w''' \Vdash A^+$ and $\mathcal{M}, w''' \Vdash A^-$ hold, a contradiction.

2.2 In order to show that $\mathcal{M}, w' \not\Vdash A^{\oplus}$ it suffices to note that $\mathcal{M}, w' \Vdash A^{-}$, which contradicts the definition of $\mathcal{M}, w' \Vdash A^{\oplus}$, given that accessibility is reflexive, *i.e.* $w' \leq w'$.

□

Lemma 11 (Non-contradiction of forcing)**.** *If $\mathcal{M}, w \Vdash P$ then $\mathcal{M}, w \not\Vdash P^{\sim}$.*

Proof. Suppose that both $\mathcal{M}, w \Vdash P$ and $\mathcal{M}, w \Vdash P^{\sim}$ hold. By stabilization (Lem. 10) there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash P$ or $\mathcal{M}, w' \Vdash P^{\sim}$ but *not both.* However, by monotonicity (Lem. 9) we know that both $\mathcal{M}, w' \Vdash P$ and $\mathcal{M}, w' \Vdash P^{\sim}$ must hold, a contradiction. □

Lemma 12 (Rule of classical forcing)**.**

1. *($\mathcal{M}, w \Vdash A^{\oplus}$) if and only if, for all $w' \geq w$, ($\mathcal{M}, w' \Vdash A^{\ominus}$) implies ($\mathcal{M}, w' \Vdash A^{+}$).*
2. *($\mathcal{M}, w \Vdash A^{\ominus}$) if and only if, for all $w' \geq w$, ($\mathcal{M}, w' \Vdash A^{\oplus}$) implies ($\mathcal{M}, w' \Vdash A^{-}$).*

Proof. We only prove the first item. The second one is symmetric, flipping all the signs.

($\Rightarrow$) Suppose that $\mathcal{M}, w \Vdash A^{\oplus}$, let $w' \geq w$, and let us show that the implication $(\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^{+})$ holds. In fact, the implication holds vacuously, given that $\mathcal{M}, w' \Vdash A^{\oplus}$ by monotonicity (Lem. 9), and therefore $\mathcal{M}, w' \not\Vdash A^{\ominus}$ by non-contradiction (Lem. 11).

($\Leftarrow$) Suppose that for every $w' \geq w$ the implication $(\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^{+})$ holds. Let us show that $\mathcal{M}, w \Vdash A^{\oplus}$ holds, *i.e.* that for every $w' \geq w$ we have that $\mathcal{M}, w' \not\Vdash A^{-}$. Let w' be a world such that $w' \geq w$ and, by contradiction, suppose that $\mathcal{M}, w' \Vdash A^{-}$. Then by non-contradiction (Lem. 11) we have that $\mathcal{M}, w' \not\Vdash A^{+}$. Hence, to obtain a contradiction, using the implication of the hypothesis, it suffices to show that $\mathcal{M}, w' \Vdash A^{\ominus}$, that is, that for every $w'' \geq w'$ we have that $\mathcal{M}, w'' \not\Vdash A^{+}$. Indeed, let $w'' \geq w'$. By monotonicity (Lem. 9) $\mathcal{M}, w'' \Vdash A^{-}$, so by non-contradiction (Lem. 11) $\mathcal{M}, w'' \not\Vdash A^{+}$, as required.

□

Proposition 13 (Soundness)**.** *If $\Gamma \vdash P$ is provable in* PRK*, then $\Gamma \Vvdash P$.*

Proof. By induction on the derivation of $\Gamma \vdash P$. The axiom rule, and the introduction and elimination rules for conjunction, disjunction, and negation are straightforward using the definition of Kripke model. The interesting cases are the following rules:

- **Abs**: let $\Gamma \vdash Q$ be derived from $\Gamma \vdash A^{+}$ and $\Gamma \vdash A^{-}$. Suppose that $\mathcal{M}, w \Vvdash \Gamma$ holds in an arbitrary world w under an arbitrary Kripke model $\mathcal{M}$, and let us show that $\mathcal{M}, w \Vdash Q$. Note that by IH we have that $\mathcal{M}, w \Vdash A^{+}$ and $\mathcal{M}, w \Vdash A^{-}$. But this is impossible by non-contradiction (Lem. 11). Hence $\mathcal{M}, w \Vdash Q$.

- **IC^{+}**: let $\Gamma \vdash A^{\oplus}$ be derived from $\Gamma, A^{\ominus} \vdash A^{+}$. Suppose that $\mathcal{M}, w \Vvdash \Gamma$ holds in an arbitrary world w under an arbitrary Kripke model $\mathcal{M}$, and let us show that $\mathcal{M}, w \Vdash A^{\oplus}$. We claim that for every $w' \geq w$ the implication $(\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^{+})$ holds. Indeed, suppose that $\mathcal{M}, w' \Vdash A^{\ominus}$. Moreover, by monotonicity (Lem. 9), we have that $\mathcal{M}, w' \Vdash\!\!\!\vdash \Gamma$. So $\mathcal{M}, w' \Vdash\!\!\!\vdash \Gamma, A^{\ominus}$ holds. Hence by IH we have that $\mathcal{M}, w' \Vdash A^{+}$. Given that the implication $(\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^{+})$ holds for all $w' \geq w$, using the rule of classical forcing (Lem. 12) we conclude that $\mathcal{M}, w \Vdash A^{\oplus}$, as required.

- **IC^{-}**: similar to the IC^{+} case.
- **EC^{+}**: similar to the Abs case.
- **EC^{-}**: similar to the Abs case.

□

3 Canonical proof of completeness

To prove **completeness**, we follow the standard methodology, which proceeds by contraposition assuming that $\Gamma \nvdash P$ and building a counter-model. The counter-model is given by a Kripke model $\mathcal{M}_0$ and a world w such that $\mathcal{M}_0, w \Vdash\!\!\!\vdash \Gamma$ but $\mathcal{M}_0, w \nVdash\!\!\!\vdash P$. In fact, the choice of the Kripke model $\mathcal{M}_0$ does not depend on Γ nor P. Rather, $\mathcal{M}_0$ is always chosen to be the *canonical* Kripke model whose worlds are *saturated* sets of propositions (*prime theories*, sometimes called *disjunctive theories*). Completeness is obtained by taking Γ and *saturating* it it to a prime theory Γ' which then verifies $\mathcal{M}_0, \Gamma' \Vdash\!\!\!\vdash \Gamma$ but $\mathcal{M}_0, \Gamma' \nVdash\!\!\!\vdash P$.

Definition 14 (Prime theory)**.** A *prime theory* is a possibly infinite set of propositions Γ such that the following hold:

1. **Closure by deduction.** If $\Gamma \vdash P$ then $P \in \Gamma$.
2. **Consistency.** Γ is consistent. Equivalently, there exists P such that $P \notin \Gamma$.
3. **Disjunctive property.**
   - If $(A \vee B)^{+} \in \Gamma$ then either $A^{\oplus} \in \Gamma$ or $B^{\oplus} \in \Gamma$.
   - If $(A \wedge B)^{-} \in \Gamma$ then either $A^{\ominus} \in \Gamma$ or $B^{\ominus} \in \Gamma$.

Lemma 15 (Saturation)**.** *Let Γ be a consistent set of propositions, and let Q be a proposition such that $\Gamma \nvdash Q$. Then there exists a prime theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash Q$.*

Proof. Consider an enumeration of all propositions $(P_1, P_2, \ldots)$. We build a sequence of consistent sets $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$, with the invariant that $\Gamma_n \nvdash Q$ for all $n \geq 0$, according to the following construction.

In the n-th step, suppose that $\Gamma_1, \ldots, \Gamma_n$ have already been constructed, and consider the first proposition P in the enumeration such that $\Gamma_n \vdash P$ but the disjunctive property fails for P, that is, either P is of the form $(A \vee B)^{+}$ with $A^{\oplus}, B^{\oplus} \notin \Gamma_n$ or P is of the form $(A \wedge B)^{-}$ with $A^{\ominus}, B^{\ominus} \notin \Gamma_n$. There are two subcases:

1. If $P = (A \vee B)^{+}$ with $A^{\oplus}, B^{\oplus} \notin \Gamma_n$, note that $\Gamma_n, A^{\oplus} \vdash Q$ and $\Gamma_n, B^{\oplus} \vdash Q$ cannot both hold simultaneously. Indeed, if both $\Gamma_n, A^{\oplus} \vdash Q$ and $\Gamma_n, B^{\oplus} \vdash Q$ hold, given that also $\Gamma_n \vdash (A \vee B)^{+}$, applying E$\vee^{+}$ we would have $\Gamma_n \vdash Q$, contradicting the hypothesis. Hence we may define Γ_{n+1} as follows:

$$\Gamma_{n+1} \stackrel{\text{def}}{=} \begin{cases} \Gamma_n \cup \{A^{\oplus}\} & \text{if } \Gamma_n, A^{\oplus} \nvdash Q \\ \Gamma_n \cup \{B^{\oplus}\} & \text{otherwise} \end{cases}$$

   Note that, in the second case, $\Gamma_n, B^{\oplus} \nvdash Q$ holds, and that Γ_{n+1} is still consistent.

2. If $P = (A \wedge B)^-$ with $A^\ominus, B^\ominus \notin \Gamma_n$, the construction is similar, defining Γ_{n+1} as either $\Gamma_n \cup \{A^\ominus\}$ or $\Gamma_n \cup \{B^\ominus\}$.

Now we define Γ_ω and Γ' as follows:

$$\begin{array}{rcl} \Gamma_\omega & \stackrel{\text{def}}{=} & \bigcup_{n\in\mathbb{N}} \Gamma_n \\ \Gamma' & \stackrel{\text{def}}{=} & \Gamma_\omega \cup \{A^\pm \mid \Gamma_\omega \vdash A^\pm\} \end{array}$$

Note that $\Gamma \subseteq \Gamma_\omega \subseteq \Gamma'$. Moreover, we claim that Γ' is a prime theory:

- **Closure by deduction**: Let $\Gamma' \vdash P$, and let us show that $P \in \Gamma'$. Since all propositions in Γ' of the form $A^\pm$ are provable from Γ_ω, this means that $\Gamma_\omega \vdash P$ by the Cut rule (Lem. 2). We consider four subcases, depending on the mode of P. We only study the positive cases; the negative cases are symmetric:
  1. *Strong proof,* i.e. $P = A^+$. Then $\Gamma_\omega \vdash A^+$ so $A^+ \in \Gamma'$ by definition of Γ'.
  2. *Classical proof,* i.e. $P = A^\oplus$. Then $\Gamma_\omega \vdash A^\oplus$ so in particular $\Gamma_\omega \vdash (A \vee A)^+$ applying the $\mathrm{I}\vee_1^+$ rule. Then there is an n_0 such that $\Gamma_n \vdash (A \vee A)^+$ for all $n \geq n_0$. Then it cannot be the case that $A^\oplus \notin \Gamma_n$ for all $n \geq n_0$, because the proposition $(A \vee A)^+$ must be eventually treated by the construction of $(\Gamma_n)_{n\in\mathbb{N}}$ above. This means that there is an $n \geq n_0$ such that $A^\oplus \in \Gamma_n$, and therefore $A^\oplus \in \Gamma_\omega \subseteq \Gamma'$, as required.
- **Consistency**: It suffices to note that $\Gamma' \nvdash Q$. Indeed, suppose that $\Gamma' \vdash Q$. Then $\Gamma_\omega \vdash Q$ by the cut rule (Lem. 2), so there exists an n_0 such that $\Gamma_n \vdash Q$ for all $n \geq n_0$. This contradicts the invariant of the construction of $(\Gamma_n)_{n\in\mathbb{N}}$ above.
- **Disjunctive property**: We consider only the positive case. The negative case is symmetric. Suppose that $\Gamma' \vdash (A \vee B)^+$. Then $\Gamma_\omega \vdash (A \vee B)^+$ by the cut rule (Lem. 2), so there exists an n_0 such that $\Gamma_n \vdash (A \vee B)^+$ for all $n \geq n_0$. Then it cannot be the case that $A^\oplus, B^\oplus \notin \Gamma_n$ for all $n \geq n_0$, because the proposition $(A \vee B)^+$ must be eventually treated by the construction of $(\Gamma_n)_{n\in\mathbb{N}}$ above. This means that there is an $n \geq n_0$ such that either $A^\oplus \in \Gamma_n$ or $B^\oplus \in \Gamma_n$, and therefore we have that either $A^\oplus \in \Gamma_\omega \subseteq \Gamma'$, or $B^\oplus \in \Gamma_\omega \subseteq \Gamma'$, as required.

Finally, note that $\Gamma' \nvdash Q$, as has already been shown in the proof of consistency above. □

In the following lemma we use an encoding of falsity with the pure proposition $\bot \stackrel{\text{def}}{=} (\alpha_0 \wedge \neg\alpha_0)$ for some fixed propositional variable α_0. Remark that $\Gamma \vdash \bot^\ominus$ is provable, being an instance of the law of non-contradiction (Ex. 5).

Lemma 16 (Consistent extension)**.** *Let Γ be a consistent set, and let P be a proposition. Then $\Gamma \cup \{P\}$ and $\Gamma \cup \{P^\sim\}$ are not both inconsistent.*

Proof. Suppose that $\Gamma \cup \{P\}$ and $\Gamma \cup \{P^\sim\}$ are both inconsistent. In particular we have that $\Gamma, P \vdash \bot^\oplus$ and $\Gamma, P^\sim \vdash \bot^\oplus$. By the projection lemma (Lem. 3) we have that $\Gamma, \bigcirc P \vdash \bot^\oplus$ and $\Gamma, \bigcirc P^\sim \vdash \bot^\oplus$. Moreover, by contraposition (Lem. 2) we have that $\Gamma, \bot^\ominus \vdash \bigcirc P^\sim$ and $\Gamma, \bot^\ominus \vdash \bigcirc P$. Since $\bot^\ominus$ is provable (Ex. 5), applying the cut rule (Lem. 2) we have that $\Gamma \vdash \bigcirc P^\sim$ and $\Gamma \vdash \bigcirc P$. The generalized absurdity rule allows us to derive $\Gamma \vdash Q$ for any Q from these two sequents, so Γ is inconsistent. This contradicts the hypothesis that Γ is consistent. □

Definition 17 (Canonical model)**.** The *canonical model* is the structure $\mathcal{M}_0 = (\mathcal{W}_0, \subseteq, \mathcal{V}^+, \mathcal{V}^-)$:

1. $\mathcal{W}_0$ is the set of all prime theories, *i.e.* $\mathcal{W}_0 \stackrel{\text{def}}{=} \{\Gamma \mid \Gamma \text{ is prime}\}$.
2. $\subseteq$ is the set-theoretic inclusion between prime theories.
3. $\mathcal{V}^+_\Gamma = \{\alpha \mid \alpha^+ \in \Gamma\}$ and $\mathcal{V}^-_\Gamma = \{\alpha \mid \alpha^- \in \Gamma\}$.

Lemma 18. *The canonical model is a Kripke model.*

Proof. Let us check the two required properties. **Monotonicity** is immediate, since if $\Gamma \subseteq \Gamma'$ then $\alpha^\pm \in \Gamma$ implies $\alpha^\pm \in \Gamma'$. For **stabilization**, let Γ be a prime theory and let α be a propositional variable. First note that $\Gamma \cup \{\alpha^+\}$ and $\Gamma \cup \{\alpha^-\}$ cannot both be inconsistent, by the consistent extension lemma (Lem. 16). We consider two subcases, depending on whether $\Gamma \cup \{\alpha^+\}$ is consistent:

1. *If* $\Gamma \cup \{\alpha^+\}$ *is consistent.* Then $\Gamma, \alpha^+ \nvdash \alpha^-$ because $\Gamma, \alpha^+ \vdash \alpha^-$ would make the set $\Gamma \cup \{\alpha^+\}$ inconsistent. Then by saturation (Lem. 15) there is a prime theory $\Gamma' \supseteq \Gamma \cup \{\alpha^+\}$ such that $\Gamma' \nvdash \alpha^-$. Hence we have that $\Gamma' \supseteq \Gamma$ with $\alpha \in \mathcal{V}^+_{\Gamma'} \setminus \mathcal{V}^-_{\Gamma'}$.
2. *Otherwise, so* $\Gamma \cup \{\alpha^-\}$ *is consistent.* Similarly as in the previous case, we have that $\Gamma, \alpha^- \nvdash \alpha^+$, so by saturation (Lem. 15) there is a prime theory $\Gamma' \supseteq \Gamma \cup \{\alpha^-\}$ such that $\Gamma' \nvdash \alpha^+$, and this implies that $\alpha \in \mathcal{V}^-_{\Gamma'} \setminus \mathcal{V}^+_{\Gamma'}$.

$\square$

Lemma 19 (Main Semantic Lemma)**.** *Let* Γ *be a prime theory. Then* $\mathcal{M}_0, \Gamma \Vdash P$ *holds in the canonical model if and only if* $P \in \Gamma$.

Proof. We proceed by induction on the measure $\#(P)$. We only study the positive cases, the negative cases are symmetric.

- **Propositional variable** ($P = \alpha^+$):

$$\mathcal{M}_0, \Gamma \Vdash \alpha^+ \iff \alpha \in \mathcal{V}^+_\Gamma \iff \alpha^+ \in \Gamma$$

- **Strong conjunction** ($P = (A \wedge B)^+$):

$$\begin{array}{lll} & \mathcal{M}_0, \Gamma \Vdash (A \wedge B)^+ & \\ \iff & \mathcal{M}_0, \Gamma \Vdash A^\oplus \text{ and } \mathcal{M}_0, \Gamma \Vdash B^\oplus & \\ \iff & A^\oplus \in \Gamma \text{ and } B^\oplus \in \Gamma & \text{by IH} \\ \iff & (A \wedge B)^+ \in \Gamma & \end{array}$$

  The last equivalence uses the fact that Γ is closed by deduction, using rule $\mathrm{I}\wedge^+$ for the "only if" direction and rules $\mathrm{E}\wedge^+_1, \mathrm{E}\wedge^+_2$ for the "if" direction.

- **Strong disjunction** ($P = (A \vee B)^+$):

$$\begin{array}{lll} & \mathcal{M}_0, \Gamma \Vdash (A \vee B)^+ & \\ \iff & \mathcal{M}_0, \Gamma \Vdash A^\oplus \text{ or } \mathcal{M}_0, \Gamma \Vdash B^\oplus & \\ \iff & A^\oplus \in \Gamma \text{ or } B^\oplus \in \Gamma & \text{by IH} \\ \iff & (A \vee B)^+ \in \Gamma & \end{array}$$

  The last equivalence uses the fact that Γ is a prime theory, using rules $\mathrm{I}\vee^+_1$ and $\mathrm{I}\vee^+_2$ for the "only if" direction, and the fact that Γ is disjunctive for the "if" direction.

- **Strong negation** ($P = (\neg A)^+$):

$$\begin{array}{rcll} \mathcal{M}_0, \Gamma \Vdash (\neg A)^+ & \iff & \mathcal{M}_0, \Gamma \Vdash A^\ominus & \\ & \iff & A^\ominus \in \Gamma & \text{by IH} \\ & \iff & (\neg A)^+ \in \Gamma & \end{array}$$

The last equivalence uses the fact that Γ is closed by deduction, using rule $\mathrm{I}\neg^+$ for the "only if" direction and rule $\mathrm{E}\neg^+$ for the "if" direction.

- **Classical proposition** ($P = A^\oplus$):

$$\begin{array}{rcll} \mathcal{M}_0, \Gamma \Vdash A^\oplus & \iff & \forall \Gamma' \supseteq \Gamma,\ \mathcal{M}_0, \Gamma' \nVdash A^- & \\ & \iff & \forall \Gamma' \supseteq \Gamma,\ A^- \notin \Gamma' & \text{by IH} \\ & \iff & A^\oplus \in \Gamma & \end{array}$$

Note that Γ' does not vary over arbitrary sets of propositions, but only over prime theories. To justify the last equivalence, we prove each implication separately:

($\Rightarrow$) We show the contrapositive. Let $A^\oplus \notin \Gamma$ and let us show that there is a prime theory $\Gamma' \supseteq \Gamma$ such that $A^- \in \Gamma'$. First we claim that $\Gamma \cup \{A^-\}$ is consistent.

Proof of the claim. Suppose by contradiction that $\Gamma \cup \{A^-\}$ is inconsistent. Then in particular $\Gamma, A^- \vdash \bot^\oplus$. (Recall that we encode falsity as $\bot \stackrel{\text{def}}{=} (\alpha_0 \wedge \neg\alpha_0)$). By the projection lemma (Lem. 3) we have that $\Gamma, A^\ominus \vdash \bot^\oplus$. By contraposition (Lem. 2) $\Gamma, \bot^\ominus \vdash A^\oplus$. Since $\bot^\ominus$ is provable (Ex. 5), by the cut rule (Lem. 2) we have that $\Gamma \vdash A^\oplus$. But Γ is closed by deduction, so $A^\oplus \in \Gamma$. This contradicts the fact that $A^\oplus \notin \Gamma$ and concludes the proof of the claim.

Now since $\Gamma \cup \{A^-\}$ is consistent, by saturation (Lem. 15), we may extend it to a prime theory $\Gamma' \supseteq \Gamma \cup \{A^-\}$. This concludes this case.

($\Leftarrow$) Suppose that $A^\oplus \in \Gamma$, and let $\Gamma' \supseteq \Gamma$ such that $A^- \in \Gamma'$. Then since Γ' is closed by deduction, using the IC^+ rule we have that $A^\ominus \in \Gamma'$. Since Γ' contains both $A^\oplus$ and $A^\ominus$, using the generalized absurdity rule we may derive an arbitrary proposition from Γ', which means that Γ' is inconsistent, contradicting the fact that Γ' is a prime theory.

□

Theorem 20 (Completeness)**.** *If* $\Gamma \Vvdash P$ *then* $\Gamma \vdash P$.

Proof. The proof is by contraposition, *i.e.* let $\Gamma \nvdash P$ and let us show that there is a Kripke model $\mathcal{M}$ and a world w such that $\mathcal{M}, w \Vvdash \Gamma$ but $\mathcal{M}, w \nVdash P$. Note that Γ is consistent, so by Saturation (Lem. 15) there exists a prime theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash P$. Note that $\mathcal{M}_0, \Gamma' \Vvdash \Gamma$ because, by the Main Semantic Lemma (Lem. 19), we have that $\mathcal{M}_0, \Gamma' \Vdash Q$ for every $Q \in \Gamma \subseteq \Gamma'$. Moreover, also by the Main Semantic Lemma (Lem. 19), we have that $\mathcal{M}_0, \Gamma' \nVdash P$ because $P \notin \Gamma'$. □

4 Examples and properties

Before moving away from our Kripke semantics, we show how the duality principle (Lem. 4) presented in the previous chapter translates to Kripke models. We also provide a counter example for the strong version of the law of the excluded middle.

Lemma 21 (Duality over Kripke models)**.** *Given a Kripke model $\mathcal{M} = (\mathcal{W}, \leq, \mathcal{V}^+, \mathcal{V}^-)$, a world $w \in \mathcal{W}$ and a proposition P, such that $\mathcal{M}, w \Vdash P$, then the model given by $\mathcal{M}^\perp \stackrel{\text{def}}{=} (\mathcal{W}, \leq, \mathcal{V}^-, \mathcal{V}^+)$ forces the dual of P, that is $\mathcal{M}^\perp, w \Vdash P^\perp$, over the same world w.*

Proof. We prove it by induction on the measure $\#(P)$. We only check the positive propositions; the negative cases are dual:

- **Propositional variable** ($P = \alpha^+$): Let $\mathcal{M}, w \Vdash \alpha^+$, that is $\alpha \in \mathcal{V}^+_w$. Then $\mathcal{M}^\perp, w \Vdash \alpha^-$ since $\alpha \in \mathcal{V}^+_w$ and $\mathcal{V}^+$ is the negative set of propositional variables of $\mathcal{M}^\perp$.
- **Conjunction** ($P = (A \wedge B)^+$): Let $\mathcal{M}, w \Vdash (A \wedge B)^+$, that is $\mathcal{M}, w \Vdash A^\oplus$ and $\mathcal{M}, w \Vdash B^\oplus$. Then by IH $\mathcal{M}^\perp, w \Vdash (A^\perp)^\ominus$ and $\mathcal{M}^\perp, w \Vdash (B^\perp)^\ominus$ so $\mathcal{M}^\perp, w \Vdash (A^\perp \vee B^\perp)^-$.
- **Disjunction** ($P = (A_1 \vee A_2)^+$): Let $\mathcal{M}, w \Vdash (A_1 \vee A_2)^+$, that is $\mathcal{M}, w \Vdash {A_i}^\oplus$ for some $i \in \{1, 2\}$. Then by IH $\mathcal{M}^\perp, w \Vdash (A_i^\perp)^\ominus$ so $\mathcal{M}^\perp, w \Vdash (A_1^\perp \wedge A_2^\perp)^-$.
- **Negation** ($P = (\neg A)^+$): Let $\mathcal{M}, w \Vdash (\neg A)^+$, that is $\mathcal{M}, w \Vdash A^\ominus$. Then by IH $\mathcal{M}^\perp, w \Vdash (A^\perp)^\oplus$ so $\mathcal{M}^\perp, w \Vdash (\neg A^\perp)^-$.
- **Classical proposition** ($P = A^\oplus$): Let $\mathcal{M}, w \Vdash A^\oplus$, that is, for every $w' \geq w$ we have that $\mathcal{M}, w' \nVdash A^-$. Our goal is to prove that $\mathcal{M}^\perp, w \Vdash (A^\perp)^\ominus$, which is true if there is no $w' \geq w$ such that $\mathcal{M}^\perp, w' \Vdash (A^\perp)^+$. So assume there is $w'' \geq w$ such that $\mathcal{M}^\perp, w'' \Vdash (A^\perp)^+$ by IH and the fact that $\mathcal{M}^{\perp\perp} = \mathcal{M}$, this would mean that $\mathcal{M}, w'' \Vdash A^-$, which would violate the hypothesis. Therefore, $\mathcal{M}^\perp, w \Vdash (A^\perp)^\ominus$.

□

Corollary 22 (Duality Principle (Kripke proof))**.** *From the previous lemma a trivial corollary can be extracted, equivalent to Lem. 4, but proved using the Kripke models semantic.*

If $P_1, \ldots, P_n \Vvdash Q$ then $P_1^\perp, \ldots, P_n^\perp \Vvdash Q^\perp$.

Example 23 (Law of excluded middle)**.** *Similarly to Ex. 5, we prove that the law of excluded middle holds in the Kripke semantics presented in this chapter. That is, $\Vvdash (A \vee \neg A)^\oplus$.*

To prove it, we need to show that for every model $\mathcal{M}$ and every world w we have that $\mathcal{M}, w \Vdash (A \vee \neg A)^\oplus$, which is essentially the same as saying that for every world w we have that $\mathcal{M}, w \nVdash (A \vee \neg A)^-$.

By contradiction, assume there is a world w such that $\mathcal{M}, w \Vdash (A \vee \neg A)^-$, that is $\mathcal{M}, w \Vdash A^\ominus$ and $\mathcal{M}, w \Vdash \neg A^\ominus$.

From the first fact, we know that $\mathcal{M}, w' \nVdash A^+$ holds in every world $w' \geq w$. From the second fact, we know that $\mathcal{M}, w'' \nVdash \neg A^+$ holds in every world $w'' \geq w$, that means that $\mathcal{M}, w'' \nVdash A^\ominus$. Hence there exists a world $w''' \geq w'' \geq w$ such that $\mathcal{M}, w''' \Vdash A^+$.

Since $w''' \geq w$, we have that both $\mathcal{M}, w''' \nVdash A^+$ (Lem. 9) and $\mathcal{M}, w''' \Vdash A^+$, reaching our beloved absurdity.

Example 24 (Counter-model for the strong excluded middle)**.** *There is a Kripke model $\mathcal{M}$ with a world w_0 such that $\mathcal{M}, w_0 \nVdash (\alpha \vee \neg\alpha)^+$. Indeed, let $\mathcal{P}$ be the set of all propositional variables, and let $\mathcal{M}$ be the Kripke model such that $\mathcal{W} = \{w_0, w_1, w_2\}$ with $w_0 \leq w_1$ and $w_0 \leq w_2$, where $\mathcal{V}^+$ and $\mathcal{V}^-$ are defined as follows:*

	$\mathcal{V}^+$	$\mathcal{V}^-$
w_0	$\varnothing$	$\varnothing$
w_1	$\mathcal{P}$	$\varnothing$
w_2	$\varnothing$	$\mathcal{P}$

It is easy to verify that $\mathcal{M}$ is a Kripke model and that $\mathcal{M}, w_0 \nVdash (\alpha \vee \neg\alpha)^+$, even if the classical excluded middle does hold.

4. PROPOSITIONS AS TYPES

In this section we will provide PRK with a Curry–Howard interpretation, developing a calculus for it, dubbed λ^{PRK}. This calculus will be given a set of reduction rules, that enjoys a number of good properties, in particular subject reduction, confluence, and termination are all proved to hold in this chapter; as well as a notion of canonicity. Finally, we present an extension of λ^{PRK} with an extra reduction rule similar to η-reduction, that will be of importance in the next chapter.

Remaining true to the *propositions as types* paradigm, from this section onwards, we use the words propositions and *types* interchangeably to refer to $P, Q, \ldots$.

1 Syntax and typing

We assume given a denumerable set of variables $x, y, z, \ldots$. The set of *typing contexts* is defined by the grammar $\Gamma ::= \varnothing \mid \Gamma, x : P$, where each variable is assumed to occur at most once in a typing context. Typing contexts are considered implicitly up to reordering.

The set of terms is given by the following abstract syntax. The letter i ranges over the set $\{1, 2\}$. Some terms are decorated with a plus or a minus sign. In the grammar we write "$^{\pm}$" to stand for either "$^{+}$" or "$^{-}$".

$t, s, u, \ldots$	$::=$	x	variable
	$\mid$	$t \bowtie_P s$	absurdity
	$\mid$	$\langle t, s\rangle^{\pm}$	$\wedge^+$ / $\vee^-$ introduction
	$\mid$	$\pi_i^{\pm}(t)$	$\wedge^+$ / $\vee^-$ elimination
	$\mid$	$\mathsf{in}_i^{\pm}(t)$	$\vee^+$ / $\wedge^-$ introduction
	$\mid$	$\delta^{\pm} t\, [_{x:P}.s][_{y:Q}.u]$	$\vee^+$ / $\wedge^-$ elimination
	$\mid$	$\nu^{\pm} t$	$\neg^+$ / $\neg^-$ introduction
	$\mid$	$\mu^{\pm} t$	$\neg^+$ / $\neg^-$ elimination
	$\mid$	$\mathsf{IC}^{\pm}_{(x:P)}.\, t$	classical introduction
	$\mid$	$t \bullet^{\pm} s$	classical elimination

The notions of free and bound occurrences of variables are defined as expected considering that $\delta^{\pm} t\, [_{x:P}.s][_{y:Q}.u]$ binds occurrences of x in s and occurrences of y in u, whereas $\mathsf{IC}^{\pm}_{x:P}.\, t$ binds occurrences of x in t. We work implicitly modulo α-renaming of bound variables. We write $\mathsf{fv}(t)$ for the set of free variables of t, and $t[x := s]$ for the capture-avoiding substitution of x by s in t.

Sometimes we omit type decorations if they are irrelevant or clear from the context, for example, we may write $\mathsf{IC}^{+}_{x}.\, t$ rather than $\mathsf{IC}^{+}_{(x:A^{\ominus})}.\, t$, and $t \bowtie s$ rather than $t \bowtie_P s$. Sometimes we also omit the name of unused bound variables, writing "_" instead; *e.g.* if $x \notin \mathsf{fv}(t)$ we may write $\mathsf{IC}^{+}_{\_}.\, t$ rather than $\mathsf{IC}^{+}_{(x:A^{\ominus})}.\, t$.

Definition 25 (The λ^{PRK} type system)**.** Typing judgments are of the form $\Gamma \vdash t : P$. Derivability of judgments is defined inductively by the following typing rules:

$$\frac{}{\Gamma, x : P \vdash x : P}\ \text{Ax} \qquad \frac{\Gamma \vdash t : A^{+} \quad \Gamma \vdash s : A^{-}}{\Gamma \vdash t \bowtie_Q s : Q}\ \text{Abs}$$

$$\frac{\Gamma \vdash t : A^{\oplus} \quad \Gamma \vdash s : B^{\oplus}}{\Gamma \vdash \langle t, s\rangle^{+} : (A \wedge B)^{+}}\ \mathrm{I}\wedge^{+} \qquad \frac{\Gamma \vdash t : A^{\ominus} \quad \Gamma \vdash s : B^{\ominus}}{\Gamma \vdash \langle t, s\rangle^{-} : (A \vee B)^{-}}\ \mathrm{I}\vee^{-}$$

$$\frac{\Gamma \vdash t : (A_1 \wedge A_2)^{+} \quad i \in \{1,2\}}{\Gamma \vdash \pi_i^{+}(t) : A_i{}^{\oplus}}\ \mathrm{E}\wedge_i^{+}$$

$$\frac{\Gamma \vdash t : (A_1 \vee A_2)^{-} \quad i \in \{1,2\}}{\Gamma \vdash \pi_i^{-}(t) : A_i{}^{\ominus}}\ \mathrm{E}\vee_i^{-}$$

$$\frac{\Gamma \vdash t : A_i{}^{\oplus} \quad i \in \{1,2\}}{\Gamma \vdash \mathsf{in}_i^{+}(t) : (A_1 \vee A_2)^{+}}\ \mathrm{I}\vee_i^{+} \qquad \frac{\Gamma \vdash t : A_i{}^{\ominus} \quad i \in \{1,2\}}{\Gamma \vdash \mathsf{in}_i^{-}(t) : (A_1 \wedge A_2)^{-}}\ \mathrm{I}\wedge_i^{-}$$

$$\frac{\Gamma \vdash t : (A \vee B)^{+} \quad \Gamma, x : A^{\oplus} \vdash s : P \quad \Gamma, y : B^{\oplus} \vdash u : P}{\Gamma \vdash \delta^{+} t\, [_{x:A^{\oplus}}.s][_{y:B^{\oplus}}.u] : P}\ \mathrm{E}\vee^{+}$$

$$\frac{\Gamma \vdash t : (A \wedge B)^{-} \quad \Gamma, x : A^{\ominus} \vdash s : P \quad \Gamma, y : B^{\ominus} \vdash u : P}{\Gamma \vdash \delta^{-} t\, [_{x:A^{\ominus}}.s][_{y:B^{\ominus}}.u] : P}\ \mathrm{E}\wedge^{-}$$

$$\frac{\Gamma \vdash t : A^{\ominus}}{\Gamma \vdash \nu^{+} t : (\neg A)^{+}}\ \mathrm{I}\neg^{+} \qquad \frac{\Gamma \vdash t : A^{\oplus}}{\Gamma \vdash \nu^{-} t : (\neg A)^{-}}\ \mathrm{I}\neg^{-}$$

$$\frac{\Gamma \vdash t : (\neg A)^{+}}{\Gamma \vdash \mu^{+} t : A^{\ominus}}\ \mathrm{E}\neg^{+} \qquad \frac{\Gamma \vdash t : (\neg A)^{-}}{\Gamma \vdash \mu^{-} t : A^{\oplus}}\ \mathrm{E}\neg^{-}$$

$$\frac{\Gamma, x : A^{\ominus} \vdash t : A^{+}}{\Gamma \vdash \mathsf{IC}^{+}_{(x:A^{\ominus})}.\, t : A^{\oplus}}\ \mathrm{IC}^{+} \qquad \frac{\Gamma, x : A^{\oplus} \vdash t : A^{-}}{\Gamma \vdash \mathsf{IC}^{+}_{(x:A^{\oplus})}.\, t : A^{\ominus}}\ \mathrm{IC}^{-}$$

$$\frac{\Gamma \vdash t : A^{\oplus} \quad \Gamma \vdash s : A^{\ominus}}{\Gamma \vdash t \bullet^{+} s : A^{+}}\ \mathrm{EC}^{+} \qquad \frac{\Gamma \vdash t : A^{\ominus} \quad \Gamma \vdash s : A^{\oplus}}{\Gamma \vdash t \bullet^{-} s : A^{-}}\ \mathrm{EC}^{-}$$

Remark 26. Each typing rule in λ^{PRK} (Def. 25) corresponds exactly to the rule of the same name in PRK (Def. 1). It is immediate to show that $P_1, \dots, P_n \vdash Q$ is derivable in PRK if and only if $x_1 : P_1, \dots, x_n : P_n \vdash t : Q$ is derivable in λ^{PRK} for some term t. Note, however, that in the *if* direction, there may be multiple occurrences of the same hypothesis in λ^{PRK}, indexed by different variable names, which become a single hypothesis in PRK (given that contexts of assumptions in PRK are sets and not multisets). This means that this correspondence is not, strictly speaking, an isomorphism.

1.1 Examples and properties

We begin by studying properties of λ^{PRK} from the *logical* point of view, as a type system. In particular, the following lemma adapts some of the results in Lem. 2 and Ex. 5 to λ^{PRK}, providing explicit proof terms for derivations.

Lemma 27. *The following rules are admissible in* λ^{PRK}*:*

1. **Weakening** *(W): If* $\Gamma \vdash t : P$ *and* $x \notin \mathsf{fv}(t)$ *then* $\Gamma, x : Q \vdash t : P$.
2. **Cut** *(*Cut*): if* $\Gamma, x : P \vdash t : Q$ *and* $\Gamma \vdash s : P$ *then* $\Gamma \vdash t[x:=s] : Q$.

3. **General absurdity** *(*ABS′*): if* $\Gamma \vdash t : P$ *and* $\Gamma \vdash s : P^{\sim}$*, where* P *is not necessarily strong, there is a term* $t \bowtie_Q s$ *such that* $\Gamma \vdash t \bowtie_Q s : Q$*.*
4. **Projection of conclusions** *(*PC*): if* $\Gamma \vdash t : P$ *there is a term* $\bigcirc t$*, such that* $\Gamma \vdash \bigcirc t : \bigcirc P$*.*
5. **Injection of premises** *(*IP*): if* $\Gamma, x : \bigcirc Q \vdash t : P$ *then* $\Gamma, x : Q \vdash t[x := \bigcirc x] : P$*.*
6. **Contraposition** *(*CONTRA*): if* P *is classical and* $\Gamma, x : P \vdash t : Q$*, there is a term* $\updownarrow^y_x (t)$ *such that* $\Gamma, y : Q^{\sim} \vdash \updownarrow^y_x (t) : P^{\sim}$*.*
7. **Excluded middle***: there is a term* $\pitchfork^+_A$ *such that* $\vdash \pitchfork^+_A : (A \vee \neg A)^{\oplus}$*.*
8. **Non-contradiction***: there is a term* $\pitchfork^-_A$ *such that* $\vdash \pitchfork^-_A : (A \wedge \neg A)^{\ominus}$*.*

Proof. **Weakening** and **cut** are routine by induction on the derivation of the first premise of the rule.

For **general absurdity**, it suffices to take:

$$t \bowtie_Q s \stackrel{\text{def}}{=} \begin{cases} t \blacktriangleright\!\blacktriangleleft_Q s & \text{if } P = A^+ \\ s \blacktriangleright\!\blacktriangleleft_Q t & \text{if } P = A^- \\ (t \bullet^+ s) \blacktriangleright\!\blacktriangleleft_Q (s \bullet^- t) & \text{if } P = A^{\oplus} \\ (s \bullet^+ t) \blacktriangleright\!\blacktriangleleft_Q (t \bullet^- s) & \text{if } P = A^{\ominus} \end{cases}$$

For **projection of conclusions**, consider the following transformation:

$$\bigcirc t \stackrel{\text{def}}{=} \begin{cases} t & \text{if } P = A^{\oplus} \\ t & \text{if } P = A^{\ominus} \\ \mathsf{IC}^+_{\_}.\, t & \text{if } P = A^+ \\ \mathsf{IC}^-_{\_}.\, t & \text{if } P = A^- \end{cases}$$

For **injection of premises**, it's easy to check by induction on the derivation of the first judgment that $\Gamma, x : Q \vdash t[x := \bigcirc x] : P$ is a valid judgment.

For **contraposition**, it suffices to take:

$$\updownarrow^y_x (t) \stackrel{\text{def}}{=} \begin{cases} \mathsf{IC}^-_{(x:A^{\oplus})}.\, (t \bowtie_{A^-} y) & \text{if } P = A^{\oplus} \\ \mathsf{IC}^+_{(x:A^{\ominus})}.\, (t \bowtie_{A^+} y) & \text{if } P = A^{\ominus} \end{cases}$$

For **excluded middle**, it suffices to take:

$$\begin{aligned} \pitchfork^+_A &\stackrel{\text{def}}{=} \mathsf{IC}^+_{(x:(A\vee\neg A)^{\ominus})}.\, \mathsf{in}^+_2(\mathsf{IC}^+_{(y:\neg A^{\ominus})}.\, \nu^+\pi^-_1(x \bullet^- \Delta^+_{y,A})) \\ \Delta^+_{y,A} &\stackrel{\text{def}}{=} \mathsf{IC}^+_{(\_:(A\vee\neg A)^{\ominus})}.\, \mathsf{in}^+_1(\mathsf{IC}^+_{(z:A^{\ominus})}.\, (y \bowtie_{A^+} \mathsf{IC}^+_{(\_:\neg A^{\ominus})}.\, \nu^+ z)) \end{aligned}$$

Dually, for **non-contradiction**:

$$\begin{aligned} \pitchfork^-_A &\stackrel{\text{def}}{=} \mathsf{IC}^-_{(x:(A\wedge\neg A)^{\oplus})}.\, \mathsf{in}^-_2(\mathsf{IC}^-_{(y:\neg A^{\oplus})}.\, \nu^-\pi^+_1(x \bullet^+ \Delta^-_{y,A})) \\ \Delta^-_{y,A} &\stackrel{\text{def}}{=} \mathsf{IC}^-_{(\_:(A\wedge\neg A)^{\oplus})}.\, \mathsf{in}^-_1(\mathsf{IC}^-_{(z:A^{\oplus})}.\, (y \bowtie_{A^-} \mathsf{IC}^-_{(\_:\neg A^{\oplus})}.\, \nu^- z)) \end{aligned}$$

□

2 Computation

We now turn to studying the *computational* properties of λ^{PRK}, provided with the following notion of reduction:

Definition 28 (The λ^{PRK}-calculus)**.** Typable terms of λ^{PRK} are endowed with the following rewriting rules, closed under arbitrary contexts.

$$\begin{array}{rcll}
\pi_i^{\pm}(\langle t_1, t_2\rangle^{\pm}) & \xrightarrow{\mathsf{proj}} & t_i & \text{if } i \in \{1,2\} \\
\delta^{\pm}(\mathsf{in}_i^{\pm}(t))\,[_x.s_1][_x.s_2] & \xrightarrow{\mathsf{case}} & s_i[x:=t] & \text{if } i \in \{1,2\} \\
\mu^{\pm}(\nu^{\pm}t) & \xrightarrow{\mathsf{neg}} & t & \\
(\mathsf{IC}_x^{\pm}.\,t) \bullet^{\pm} s & \xrightarrow{\mathsf{beta}} & t[x:=s] & \\
\langle t_1, t_2\rangle^{+} \bowtie \mathsf{in}_i^{-}(s) & \xrightarrow{\mathsf{absPairInj}} & t_i \bowtie s & \text{if } i \in \{1,2\} \\
\mathsf{in}_i^{+}(t) \bowtie \langle s_1, s_2\rangle^{-} & \xrightarrow{\mathsf{absInjPair}} & t \bowtie s_i & \text{if } i \in \{1,2\} \\
(\nu^{+}t) \bowtie (\nu^{-}s) & \xrightarrow{\mathsf{absNeg}} & t \bowtie s &
\end{array}$$

If many occurrences of "±" appear in the same expression, they are all supposed to stand for the same sign (either + or −).

Example 29. *If* $x : A^{\ominus} \vdash t : A^{+}$ *and* $y : A^{\oplus} \vdash s : A^{-}$ *then:*

$$\begin{array}{rl}
& (\nu^{+}(\mathsf{IC}_y^{-}.\,s)) \bowtie (\nu^{-}(\mathsf{IC}_x^{+}.\,t)) \\
\longrightarrow & (\mathsf{IC}_y^{-}.\,s) \bowtie (\mathsf{IC}_x^{+}.\,t) \\
= & ((\mathsf{IC}_x^{+}.\,t) \bullet^{+} (\mathsf{IC}_y^{-}.\,s)) \bowtie ((\mathsf{IC}_y^{-}.\,s) \bullet^{-} (\mathsf{IC}_x^{+}.\,t)) \\
\longrightarrow & t[x:=(\mathsf{IC}_y^{-}.\,s)] \bowtie ((\mathsf{IC}_y^{-}.\,s) \bullet^{-} (\mathsf{IC}_x^{+}.\,t)) \\
\longrightarrow & t[x:=(\mathsf{IC}_y^{-}.\,s)] \bowtie s[y:=(\mathsf{IC}_x^{+}.\,t)]
\end{array}$$

A first, direct, observation is that PRK's duality principle (Lem. 4) can be strengthened to obtain a **computational duality principle** for λ^{PRK}. Note that on Lem. 27, the duality principle can be used to derive a term for the excluded middle from non-contradiction, and vice versa.

Lemma 30. *If* $t^{\perp}$ *is the term that results from flipping all the signs in* t*, then* $\Gamma \vdash t : P$ *if and only if* $\Gamma^{\perp} \vdash t^{\perp} : P^{\perp}$*, and* $t \to s$ *if and only if* $t^{\perp} \to s^{\perp}$*.*

Proof. The proof is immediate given that all typing and reduction rules are symmetric. □

2.1 Subject reduction

The second computational property that we study is **subject reduction**, also known as *type preservation*. This fundamental property ensures that reduction is well-defined over the set of typable terms. More precisely:

Proposition 31 (Subject reduction)**.** *If* $\Gamma \vdash t : P$ *and* $t \to s$*, then* $\Gamma \vdash s : P$*.*

Proof. Recall that a context is a term $\mathtt{C}$ with a single free occurrence of a hole $\Box$, and that $\mathtt{C}\langle t\rangle$ denotes the capturing substitution of the term t into the hole of $\mathtt{C}$. Since reduction is closed under arbitrary contexts, the term on the left hand side is of the form $\mathtt{C}\langle t_0\rangle$ and it reduces to $\mathtt{C}\langle t_1\rangle$ contracting the redex t_0. We proceed by induction on the context $\mathtt{C}$ under which the rewriting step takes place.

The interesting case is when the context is empty. All other cases are easy by resorting to the IH. For example, if $\mathtt{C} = \mathsf{IC}_x^{+}.\,\mathtt{C}'$, the typing derivation is of the form:

$$\dfrac{\dfrac{\pi}{\Gamma, x : A^{\ominus} \vdash \mathsf{C}'\langle t \rangle : A^{+}}}{\Gamma \vdash \mathsf{IC}^{+}_{x}.\,\mathsf{C}'\langle t \rangle : A^{\oplus}}\;\mathrm{IC}^{+}$$

By IH, there is a derivation π' ending with $\Gamma, x : A \ \vdash \mathsf{C}'\langle s \rangle : A^{+}$, so we can build the following derivation to show that the type is preserved:

$$\dfrac{\dfrac{\pi'}{\Gamma, x : A^{\ominus} \vdash \mathsf{C}'\langle s \rangle : A^{+}}}{\Gamma \vdash \mathsf{IC}^{+}_{x}.\,\mathsf{C}'\langle s \rangle : A^{\oplus}}\;\mathrm{IC}^{+}$$

It remains to check the case when $\mathsf{C} = \Box$, *i.e.* when reduction takes place at the root of the term. We proceed by case analysis on each of the reduction rules. Note that most rules actually stand for two rules, depending on the instantiations of the signs. We write only the positive cases; if the signs are flipped the proof is symmetric. We use the admissible typing rules Cut and Abs′ (Lem. 27).

- **Projection** (proj): let $i \in \{1, 2\}$. We have:

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t_1 : A_1{}^{\oplus}} \qquad \dfrac{\pi_2}{\Gamma \vdash t_2 : A_2{}^{\oplus}}}{\Gamma \vdash \langle t_1, t_2 \rangle^{+} : (A_1 \wedge A_2)^{+}}\;\mathrm{I}\wedge^{+}}{\Gamma \vdash t_i : A_i{}^{\oplus}}\;\mathrm{E}\wedge^{+}_{i}$$

  Then:

$$\dfrac{\pi_i}{\Gamma \vdash t_i : A^{\oplus}}$$

- **Case** (case): let $i \in \{1, 2\}$. We have:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : A_i{}^{\oplus}}}{\Gamma \vdash \mathsf{in}^{+}_{i}(t) : (A_1 \vee A_2)^{+}}\;\mathrm{I}\vee^{+}_{i} \qquad \dfrac{\pi_1}{\Gamma, x : A_1{}^{\oplus} \vdash s_1 : P} \qquad \dfrac{\pi_2}{\Gamma, x : A_2{}^{\oplus} \vdash s_2 : P}}{\Gamma \vdash \delta^{+}(\mathsf{in}^{+}_{i}(t))\,[_{x}.s_1][_{x}.s_2] : P}\;\mathrm{E}\vee^{+}$$

  Then:

$$\dfrac{\dfrac{\pi_i}{\Gamma, x : A_i{}^{\oplus} \vdash s_i : P} \qquad \dfrac{\pi}{\Gamma \vdash t : A_i{}^{\oplus}}}{\Gamma \vdash s_i[x{:=}t] : P}\;\textsc{Cut}$$

- **Negation** (neg): We have:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : A^{\ominus}}}{\Gamma \vdash \nu^{+}t : (\neg A)^{+}}\;\mathrm{I}\neg^{+}}{\Gamma \vdash \mu^{+}(\nu^{+}t) : A^{\ominus}}\;\mathrm{E}\neg^{+}$$

  Then:

$$\dfrac{\pi}{\Gamma \vdash t : A^{\ominus}}$$

- **Beta** (beta): we have:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma, x : A^{\ominus} \vdash t : A^{+}}}{\Gamma \vdash \mathsf{IC}^{+}_{x}.\, t : A^{\oplus}}\ \mathrm{IC}^{+} \qquad \dfrac{\pi'}{\Gamma \vdash s : A^{\ominus}}}{\Gamma \vdash (\mathsf{IC}^{+}_{x}.\, t) \bullet^{+} s : A^{+}}\ \mathrm{EC}^{+}$$

Then:

$$\dfrac{\dfrac{\pi}{\Gamma, x : A^{\ominus} \vdash t : A^{+}} \qquad \dfrac{\pi'}{\Gamma \vdash s : A^{\ominus}}}{\Gamma \vdash t[x := s] : A^{+}}\ \textsc{Cut}$$

- **Absurdity Pair-Injection** (absPairInj): we have:

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t_1 : A_1{}^{\oplus}} \qquad \dfrac{\pi_2}{\Gamma \vdash t_2 : A_2{}^{\oplus}}}{\Gamma \vdash \langle t_1, t_2 \rangle^{+} : (A_1 \wedge A_2)^{+}}\ \mathrm{I}\wedge^{+} \qquad \dfrac{\dfrac{\pi'}{\Gamma \vdash s : A_i{}^{\ominus}}}{\Gamma \vdash \mathsf{in}^{-}_{i}(s) : (A_1 \wedge A_2)^{-}}\ \mathrm{I}\wedge^{-}_{i}}{\Gamma \vdash \langle t_1, t_2 \rangle^{+} \blacktriangleright\!\!\blacktriangleleft_{P} \mathsf{in}^{-}_{i}(s) : P}\ \textsc{Abs}$$

Then:

$$\dfrac{\dfrac{\pi_i}{\Gamma \vdash t_i : A_i{}^{\oplus}} \qquad \dfrac{\pi'}{\Gamma \vdash s : A_i{}^{\ominus}}}{\Gamma \vdash t_i \bowtie_{P} s : P}\ \textsc{Abs'}$$

- **Absurdity Injection-Pair** (absInjPair): similar to the previous case.
- **Absurdity Negation** (absNeg):

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : A^{\ominus}}}{\Gamma \vdash \nu^{+} t : (\neg A)^{+}}\ \mathrm{I}\neg^{+} \qquad \dfrac{\dfrac{\pi'}{\Gamma \vdash s : A^{\oplus}}}{\Gamma \vdash \nu^{-} s : (\neg A)^{-}}\ \mathrm{I}\neg^{-}}{\Gamma \vdash (\nu^{+} t) \blacktriangleright\!\!\blacktriangleleft_{P} (\nu^{-} s) : P}\ \textsc{Abs}$$

Then:

$$\dfrac{\dfrac{\pi}{\Gamma \vdash t : A^{\ominus}} \qquad \dfrac{\pi'}{\Gamma \vdash s : A^{\oplus}}}{\Gamma \vdash t \bowtie_{P} s : P}\ \textsc{Abs'}$$

□

2.2 Confluence

Third, the λ^{PRK}-calculus enjoys **confluence**, the basic property of a rewriting system stating that given reduction sequences $t_0 \to^{*} t_1$ and $t_0 \to^{*} t_2$ there must exist a term t_3 such that $t_1 \to^{*} t_3$ and $t_2 \to^{*} t_3$.

Proposition 32. *The λ^{PRK}-calculus is confluent.*

Proof. The rewriting system λ^{PRK} can be modeled as a higher-order rewriting system (HRS) in the sense of Nipkow[1]. This HRS is *orthogonal*, *i.e.* left-linear without critical pairs, which entails that it is confluent [10]. □

2.3 Termination

Our next goal is to prove that λ^{PRK} enjoys **strong normalization**, that is, that there are no infinite reduction sequences $t_1 \to t_2 \to t_3 \to \ldots$. To do so, we give a translation to System F extended with recursive type constraints, and show that it does not erase reduction steps.

System F Extended with Recursive Type Constraints.

We begin by recalling the extended System F and its relevant properties, as formulated by Mendler [11].

The set of *types* in the the extended System F is given by $A ::= \alpha \mid A \to A \mid \forall\alpha.A$ where $\alpha, \beta, \ldots$ are called *base types*. The set of *terms* is given by $t ::= x \mid \lambda x^A.\, t \mid t\, t \mid \lambda\alpha.\, t \mid t\, A$, where $\lambda\alpha.\, t$ is type abstraction and $t\, A$ is type application. We define the empty ($\mathbf{0}$), unit ($\mathbf{1}$), product ($A \times B$), and sum types ($A + B$) via their usual encodings in System F. For example, the product type is defined as $(A \times B) \overset{\text{def}}{=} \forall\alpha.((A \to B \to \alpha) \to \alpha)$ with a constructor $\langle t, s\rangle$ and an eliminator $\pi_i(t)$. See Appendix 2 for a more detailed description of the Extended System F.

A *type constraint* is an equation of the form $\alpha \equiv A$. The extended System F is parameterized by a set $\mathcal{C}$ of type constraints. Each set $\mathcal{C}$ of type constraints induces a notion of equivalence between types, written $A \equiv B$ and defined as the congruence generated by $\mathcal{C}$. Typing rules are those of the usual System F [12, Section 11.3] extended with a *conversion* rule:

$$\frac{\Gamma \vdash t : A \qquad A \equiv B}{\Gamma \vdash t : B}\,\textsc{Conv}$$

Variables occurring *positively* (resp. *negatively*) in a type A are written $\mathsf{p}(A)$ (resp. $\mathsf{n}(A)$) and defined as usual:

$$\begin{array}{rclrcl}
\mathsf{p}(\alpha) & \overset{\text{def}}{=} & \{\alpha\} & \mathsf{n}(\alpha) & \overset{\text{def}}{=} & \varnothing \\
\mathsf{p}(A \to B) & \overset{\text{def}}{=} & \mathsf{n}(A) \cup \mathsf{p}(B) & \mathsf{n}(A \to B) & \overset{\text{def}}{=} & \mathsf{p}(A) \cup \mathsf{n}(B) \\
\mathsf{p}(\forall\alpha.A) & \overset{\text{def}}{=} & \mathsf{p}(A) \setminus \{\alpha\} & \mathsf{n}(\forall\alpha.A) & \overset{\text{def}}{=} & \mathsf{n}(A) \setminus \{\alpha\}
\end{array}$$

A set of type constraints $\mathcal{C}$ verifies the *positivity condition* if for every type constraint $(\alpha \equiv A) \in \mathcal{C}$ and every type B such that $\alpha \equiv B$ one has that $\alpha \notin \mathsf{n}(B)$. Mendler's main result [11, Theorem 13] is:

Theorem 33 (Mendler, 1991)**.** *If $\mathcal{C}$ verifies the positivity condition, then System F extended with the recursive type constraints $\mathcal{C}$ is strongly normalizing.*

[1] It suffices to model it with a single sort ι, with constants such as $\pi_i^+ : \iota \to \iota$, $\mathsf{IC}^- : (\iota \to \iota) \to \iota$, etc., and rules such as $\delta^+(\mathsf{in}_1^+ x)\, f\, g \to f\, x$. A complete formulation can be found on the Appendix 3.

System F Extended with $\mathcal{C}_{\mathbf{pn}}$.

In this subsection, we describe a specific instance of the extended System F, given by a particular set of recursive type constraints called $\mathcal{C}_{\mathbf{pn}}$. Given that the set of base types is countably infinite, we may assume without loss of generality that, for any two types A, B in System F there are two type variables, called $\mathbf{p}_{A,B}$ and $\mathbf{n}_{A,B}$. More precisely, the set of type variables can be partitioned as $\mathbf{V} \uplus \mathbf{P} \uplus \mathbf{N}$ in such a way that the propositional variables of λ^{PRK} are identified with type variables of $\mathbf{V}$, and there are bijective mappings $(A, B) \mapsto \mathbf{p}_{A,B} \in \mathbf{P}$ and $(A, B) \mapsto \mathbf{n}_{A,B} \in \mathbf{N}$. Note that we do not forbid A and B to have occurrences of type variables in $\mathbf{P}$ and $\mathbf{N}$.[2]

The particular extension of System F that we use is given by the set of recursive type constraints $\mathcal{C}_{\mathbf{pn}}$, including the following equations for all types A, B:

$$\mathbf{p}_{A,B} \equiv (\mathbf{n}_{A,B} \to A) \qquad \mathbf{n}_{A,B} \equiv (\mathbf{p}_{A,B} \to B)$$

Next, we show this set of constraints complies with the positivity condition.

Proposition 34. *The set of type constraints $\mathcal{C}_{\mathbf{pn}}$ verifies Mendler's positivity condition*

Proof. Define the *complexity* of a type as follows:

$$\begin{array}{rcl} ||\alpha|| & \stackrel{\text{def}}{=} & 1 \quad \text{if } \alpha \in \mathbf{V} \\ ||\mathbf{p}_{A,B}|| = ||\mathbf{n}_{A,B}|| = ||A \to B|| & \stackrel{\text{def}}{=} & 1 + ||A|| + ||B|| \\ ||\forall\alpha.A|| & \stackrel{\text{def}}{=} & 1 + ||A|| \end{array}$$

Recall that $\mathsf{p}(A)$ (resp. $\mathsf{n}(A)$) stand for the set of type variables occurring positively (resp. negatively) in a given type A. Moreover, the set of type variables occurring *weakly positively* (resp. *weakly negatively*) in A is written $\mathsf{p}^{\mathtt{w}}(A)$ (resp. $\mathsf{n}^{\mathtt{w}}(A)$) and defined as follows:

$$\begin{array}{rcl} \mathsf{p}^{\mathtt{w}}(\alpha) & \stackrel{\text{def}}{=} & \{\alpha\} \quad \text{if } \alpha \in \mathbf{V} \\ \mathsf{p}^{\mathtt{w}}(\mathbf{p}_{A,B}) & \stackrel{\text{def}}{=} & \{\mathbf{p}_{A,B}\} \cup \mathsf{p}^{\mathtt{w}}(A) \cup \mathsf{n}^{\mathtt{w}}(B) \\ \mathsf{p}^{\mathtt{w}}(\mathbf{n}_{A,B}) & \stackrel{\text{def}}{=} & \{\mathbf{n}_{A,B}\} \cup \mathsf{n}^{\mathtt{w}}(A) \cup \mathsf{p}^{\mathtt{w}}(B) \\ \mathsf{p}^{\mathtt{w}}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{n}^{\mathtt{w}}(A) \cup \mathsf{p}^{\mathtt{w}}(B) \\ \mathsf{p}^{\mathtt{w}}(\forall\alpha.A) & \stackrel{\text{def}}{=} & \mathsf{p}^{\mathtt{w}}(A) \setminus \{\alpha\} \end{array}$$

$$\begin{array}{rcl} \mathsf{n}^{\mathtt{w}}(\alpha) & \stackrel{\text{def}}{=} & \varnothing \quad \text{if } \alpha \in \mathbf{V} \\ \mathsf{n}^{\mathtt{w}}(\mathbf{p}_{A,B}) & \stackrel{\text{def}}{=} & \mathsf{n}^{\mathtt{w}}(A) \cup \mathsf{p}^{\mathtt{w}}(B) \\ \mathsf{n}^{\mathtt{w}}(\mathbf{n}_{A,B}) & \stackrel{\text{def}}{=} & \mathsf{p}^{\mathtt{w}}(A) \cup \mathsf{n}^{\mathtt{w}}(B) \\ \mathsf{n}^{\mathtt{w}}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{p}^{\mathtt{w}}(A) \cup \mathsf{n}^{\mathtt{w}}(B) \\ \mathsf{n}^{\mathtt{w}}(\forall\alpha.A) & \stackrel{\text{def}}{=} & \mathsf{n}^{\mathtt{w}}(A) \setminus \{\alpha\} \end{array}$$

It is easy to check that $\mathsf{p}(A) \subseteq \mathsf{p}^{\mathtt{w}}(A)$ and $\mathsf{n}(A) \subseteq \mathsf{n}^{\mathtt{w}}(A)$ by simultaneous induction on A. It is also easy to check that if $\alpha \in \mathsf{p}^{\mathtt{w}}(A) \cup \mathsf{n}^{\mathtt{w}}(A)$ then $||\alpha|| \leq ||A||$, by induction on A. Moreover, let X, Y be types. A type A is said to be (X, Y)-*positive* if $\mathbf{p}_{X,Y} \in \mathsf{p}^{\mathtt{w}}(A)$ or $\mathbf{n}_{X,Y} \in \mathsf{n}^{\mathtt{w}}(A)$. Symmetrically, a type A is said to be (X, Y)-*negative* if $\mathbf{p}_{X,Y} \in \mathsf{n}^{\mathtt{w}}(A)$ or

[2] An alternative, perhaps cleaner, presentation would be to define types inductively as $A, B, \ldots ::= \alpha \mid \mathbf{p}_{A,B} \mid \mathbf{n}_{A,B} \mid A \to B \mid \forall\alpha.A$.

$\mathbf{n}_{X,Y} \in \mathsf{p}^{\mathtt{w}}(A)$. It is straightforward to prove the following **invariant** for the equivalence $A \equiv B$ between types induced by the recursive type constraints, by induction on the derivation of $A \equiv B$.

1. If $A \equiv B$, then A is (X, Y)-positive if and only if B is (X, Y)-positive.

2. If $A \equiv B$, then A is (X, Y)-negative if and only if B is (X, Y)-negative.

To prove Mendler's positivity condition, we must check that given any type variable α of the form $\mathbf{p}_{A,B}$ or of the form $\mathbf{n}_{A,B}$ (*i.e.* $\alpha \notin \mathbf{V}$), then whenever $\alpha \equiv C$ we have that α does not occur negatively in C. We consider two cases, depending on whether $\alpha = \mathbf{p}_{A,B}$ or $\alpha = \mathbf{n}_{A,B}$:

1. Let $\mathbf{p}_{A,B} \equiv C$ and suppose that $\mathbf{p}_{A,B} \in \mathsf{n}(C)$. Then we have that $\mathbf{p}_{A,B} \in \mathsf{n}^{\mathtt{w}}(C)$, so C is (A, B)-negative. By the invariant, $\mathbf{p}_{A,B}$ is also (A, B)-negative, so either $\mathbf{p}_{A,B} \in \mathsf{n}^{\mathtt{w}}(\mathbf{p}_{A,B})$ or $\mathbf{n}_{A,B} \in \mathsf{p}^{\mathtt{w}}(\mathbf{p}_{A,B})$. Both conditions are impossible, indeed:

   1.1 Suppose that $\mathbf{p}_{A,B} \in \mathsf{n}^{\mathtt{w}}(\mathbf{p}_{A,B})$. Then, given that $\mathbf{p}_{A,B}$ does not occur weakly negatively at the root of $\mathbf{p}_{A,B}$, so it must occur either inside A or inside B, so $||\mathbf{p}_{A,B}|| < ||\mathbf{p}_{A,B}||$, which is a contradiction.

   1.2 Suppose that $\mathbf{n}_{A,B} \in \mathsf{n}^{\mathtt{w}}(\mathbf{p}_{A,B})$. Then, again, $\mathbf{n}_{A,B}$ must occur either inside A or inside B, so $||\mathbf{n}_{A,B}|| < ||\mathbf{p}_{A,B}||$, which is a contradiction.

2. If $\mathbf{n}_{A,B} \equiv C$ then, symmetrically as above, we have that $\mathbf{n}_{A,B} \notin \mathsf{n}(C)$.

□

The set of constraints complies with the positivity condition, therefore:

Corollary 35. *System F extended with the recursive type constraints $\mathcal{C}_{\mathbf{pn}}$ is strongly normalizing.*

Proof. A corollary of Mendler's theorem and the previous proposition. □

Translating λ^{PRK} to System F Extended with $\mathcal{C}_{\mathbf{pn}}$.

We are now in conditions to define the translation from λ^{PRK} to System F extended with the set $\mathcal{C}_{\mathbf{pn}}$ of recursive type constraints.

Definition 36 (Translation of Propositions)**.** A proposition P of λ^{PRK} is translated into a type $[\![P]\!]$ of the extended System F, according to the following definition, given by induction on the *measure* $\#(P)$ (defined in Section 3):

$$\begin{array}{rclrcl}
[\![\alpha^+]\!] & \stackrel{\text{def}}{=} & \alpha & [\![\alpha^-]\!] & \stackrel{\text{def}}{=} & \alpha \to \mathbf{0} \\
[\![(A \wedge B)^+]\!] & \stackrel{\text{def}}{=} & [\![A^\oplus]\!] \times [\![B^\oplus]\!] & [\![(A \wedge B)^-]\!] & \stackrel{\text{def}}{=} & [\![A^\ominus]\!] + [\![B^\ominus]\!] \\
[\![(A \vee B)^+]\!] & \stackrel{\text{def}}{=} & [\![A^\oplus]\!] + [\![B^\oplus]\!] & [\![(A \vee B)^-]\!] & \stackrel{\text{def}}{=} & [\![A^\ominus]\!] \times [\![B^\ominus]\!] \\
[\![(\neg A)^+]\!] & \stackrel{\text{def}}{=} & \mathbf{1} \to [\![A^\ominus]\!] & [\![(\neg A)^-]\!] & \stackrel{\text{def}}{=} & \mathbf{1} \to [\![A^\oplus]\!] \\
[\![A^\oplus]\!] & \stackrel{\text{def}}{=} & \mathbf{p}_{[\![A^+]\!],[\![A^-]\!]} & [\![A^\ominus]\!] & \stackrel{\text{def}}{=} & \mathbf{n}_{[\![A^+]\!],[\![A^-]\!]}
\end{array}$$

Moreover, a typing context $\Gamma = (x_1 : P_1, \ldots, x_n : P_n)$ is translated as $[\![\Gamma]\!] \stackrel{\text{def}}{=} (x_1 : [\![P_1]\!], \ldots, x_n : [\![P_n]\!])$.

Note that the translation of propositions mimicks the equations for the realizability interpretation discussed in the introduction. In fact, the translation of $[\![A^{\oplus}]\!]$ is $\mathbf{p}_{[\![A^+]\!],[\![A^-]\!]}$, which is equivalent to $\mathbf{n}_{[\![A^+]\!],[\![A^-]\!]} \to [\![A^+]\!]$ according to the recursive type constraints in $\mathcal{C}_{\mathbf{pn}}$, and this in turn equals $[\![A^{\ominus}]\!] \to [\![A^+]\!]$, just as required. Similarly for the translation of $A^{\ominus}$. The translation of $(\neg A)^+$ is $(\mathbf{1} \to [\![A^{\ominus}]\!])$ rather than just $[\![A^{\ominus}]\!]$ for a technical reason, in order to ensure that each reduction step in λ^{PRK} is simulated by *at least one* step in the extended System F.

Definition 37 (Translation of Terms)**.** First, we define a family of terms $\vdash \mathsf{abs}_Q^P : [\![P]\!] \to [\![P^{\sim}]\!] \to [\![Q]\!]$ in the extended System F as follows, by induction on the measure $\#(P)$:

$$
\begin{array}{rcl}
\mathsf{abs}_Q^{\alpha^+} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathcal{E}_{[\![Q]\!]}(y\, x) \\
\mathsf{abs}_Q^{\alpha^-} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathcal{E}_{[\![Q]\!]}(x\, y) \\
\mathsf{abs}_Q^{(A\wedge B)^+} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \delta y\, [_z.\mathsf{abs}_Q^{A^{\oplus}}\ \pi_1(x)\, z][_z.\mathsf{abs}_Q^{B^{\oplus}}\ \pi_2(x)\, z] \\
\mathsf{abs}_Q^{(A\wedge B)^-} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \delta x\, [_z.\mathsf{abs}_Q^{A^{\ominus}}\ z\, \pi_1(y)][_z.\mathsf{abs}_Q^{B^{\ominus}}\ z\, \pi_2(x)] \\
\mathsf{abs}_Q^{(A\vee B)^+} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \delta x\, [_z.\mathsf{abs}_Q^{A^{\oplus}}\ z, \pi_1(y)][_z.\mathsf{abs}_Q^{B^{\oplus}}\ z, \pi_2(y)] \\
\mathsf{abs}_Q^{(A\vee B)^-} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \delta y\, [_z.\mathsf{abs}_Q^{A^{\ominus}}\ \pi_1(x)\, z][_z.\mathsf{abs}_Q^{B^{\ominus}}\ \pi_2(x)\, z] \\
\mathsf{abs}_Q^{(\neg A)^+} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathsf{abs}_Q^{A^{\ominus}}\ (x\star)\, (y\star) \\
\mathsf{abs}_Q^{(\neg A)^-} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathsf{abs}_Q^{A^{\oplus}}\ (x\star)\, (y\star) \\
\mathsf{abs}_Q^{A^{\oplus}} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathsf{abs}_Q^{A^+}\ (x\, y)\, (y\, x) \\
\mathsf{abs}_Q^{A^{\ominus}} & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathsf{abs}_Q^{A^-}\ (x\, y)\, (y\, x)
\end{array}
$$

where: $\mathcal{E}_A(t)$ denotes an inhabitant of A whenever t is an inhabitant of the empty type; $\delta t\, [_x.s][_x.u]$ is the eliminator of the sum type; $\pi_i(t)$ is the eliminator of the product type; and $\star$ denotes the trivial inhabitant of the unit type. Note that abs_Q^P behaves similarly to $\bowtie_Q$.

Now each typable term $\Gamma \vdash t : P$ in λ^{PRK} can be translated into a term $[\![\Gamma]\!] \vdash [\![t]\!] : [\![P]\!]$ of the extended System F as follows:

$$
\begin{array}{rcl}
[\![x]\!] & \stackrel{\text{def}}{=} & x \\
[\![t \bowtie_Q s]\!] & \stackrel{\text{def}}{=} & \mathsf{abs}_Q^{A^+}\ [\![t]\!]\, [\![s]\!] \\
 & & \quad \text{if } \Gamma \vdash t : A^+ \text{ and } \Gamma \vdash s : A^- \\
[\![\langle t, s\rangle^{\pm}]\!] & \stackrel{\text{def}}{=} & \langle [\![t]\!], [\![s]\!] \rangle \\
[\![\pi_i^{\pm}(t)]\!] & \stackrel{\text{def}}{=} & \pi_i([\![t]\!]) \\
[\![\mathsf{in}_i^{\pm}(t)]\!] & \stackrel{\text{def}}{=} & \mathsf{in}_i([\![t]\!]) \\
[\![\delta^{\pm} t\, [_{(x:P)}.s][_{(y:Q)}.u]]\!] & \stackrel{\text{def}}{=} & \delta [\![t]\!]\, [_{(x:[\![P]\!])}.[\![s]\!]][_{(y:[\![Q]\!])}.[\![u]\!]] \\
[\![\nu^{\pm} t]\!] & \stackrel{\text{def}}{=} & \lambda x^{\mathbf{1}}.\, [\![t]\!] \quad \text{where } x \notin \mathsf{fv}(t) \\
[\![\mu^{\pm} t]\!] & \stackrel{\text{def}}{=} & [\![t]\!]\, \star \\
[\![\mathsf{IC}^{\pm}_{(x:P)}.\, t]\!] & \stackrel{\text{def}}{=} & \lambda x^{[\![P]\!]}.\, [\![t]\!] \\
[\![t \bullet^{\pm} s]\!] & \stackrel{\text{def}}{=} & [\![t]\!]\, [\![s]\!]
\end{array}
$$

It is easy to check that $[\![\Gamma]\!] \vdash [\![t]\!] : [\![P]\!]$ holds in the extended System F by induction on the derivation of the judgment $\Gamma \vdash t : P$ in λ^{PRK}. Two straightforward properties of the translation are:

Lemma 38. 1. $\mathsf{fv}([\![t]\!]) = \mathsf{fv}(t)$*;* 2. $[\![t[x:=s]]\!] = [\![t]\!][x:=[\![s]\!]]$*.*

The key result is the following **simulation** lemma from which strong normalization follows:

Lemma 39. *If $t \to s$ in λ^{PRK} then $[\![t]\!] \to^+ [\![s]\!]$ in System F extended with $\mathcal{C}_{\mathbf{pn}}$.*

Proof. By case analysis on the rewriting rule used to derive the step $t \to s$. Note that showing contextual closure is immediate, so we only study the cases in which the rewriting rule is applied at the root:

- **Projection** (proj):

$$[\![\pi_i^{\pm}(\langle t_1, t_2\rangle^{\pm})]\!] = \pi_i(\langle [\![t_1]\!], [\![t_2]\!]\rangle) \to [\![t_i]\!]$$

- **Case** (case):

$$\begin{array}{ll} & [\![\delta^{\pm}\mathsf{in}_i^{\pm}(t)\,[_{(x:P)}.s_1][_{(x:Q)}.s_2]]\!] \\ = & \delta\mathsf{in}_i([\![t]\!])\,[_{(x:[\![P]\!])}.[\![s_1]\!]][_{(x:[\![Q]\!])}.[\![s_2]\!]] \\ \to & [\![s_i]\!][x:=[\![t]\!]] \\ = & [\![s_i[x:=t]]\!] \\ & \quad \text{by Lem. 38} \end{array}$$

- **Negation** (neg): $[\![\mu^{\pm}\nu^{\pm}t]\!] = (\lambda x^{\mathbf{1}}.\,[\![t]\!])\,\star \to [\![t]\!][x := \star] = [\![t]\!]$ by Lem. 38, since $x \notin \mathsf{fv}(t)$ by definition of $[\![\nu^{\pm}t]\!]$.

- **Beta** (beta): $[\![(\mathsf{IC}^{\pm}_{(x:P)}.\,t) \bullet^{\pm} s]\!] = (\lambda x^{[\![P]\!]}.\,[\![t]\!])\,[\![s]\!] \to [\![t]\!][x := [\![s]\!]] = [\![t[x := s]]\!]$ by Lem. 38.

- **Absurdity Pair-Injection** (absPairInj): Let $\vdash t_1 : A_1{}^{\oplus}$, $\vdash t_2 : A_2{}^{\oplus}$, and $\vdash s : A_i{}^{\ominus}$ for some $i \in \{1, 2\}$. Then:

$$\begin{array}{ll} & [\![\langle t_1, t_2\rangle^{+} \bowtie_P \mathsf{in}_i^{-}(s)]\!] \\ = & \mathsf{abs}_P^{(A_1\wedge A_2)^{+}}\,\langle [\![t_1]\!], [\![t_2]\!]\rangle\,\mathsf{in}_i([\![s]\!]) \\ \to^{+} & \delta\mathsf{in}_i([\![s]\!]) \\ & \quad [_{(z:[\![A_1{}^{\ominus}]\!])}.\mathsf{abs}_P^{A_1{}^{\oplus}}\,\pi_1(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\,z] \\ & \quad [_{(z:[\![A_2{}^{\ominus}]\!])}.\mathsf{abs}_P^{A_2{}^{\oplus}}\,\pi_2(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\,z] \\ & \quad \text{by definition of } \mathsf{abs}_P^{(A_1\wedge A_2)^{+}} \\ \to & \mathsf{abs}_P^{A_i{}^{\oplus}}\,\pi_i(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\,[\![s]\!] \\ \to & \mathsf{abs}_P^{A_i{}^{\oplus}}\,[\![t_i]\!]\,[\![s]\!] \\ \to^{+} & \mathsf{abs}_P^{A_i{}^{+}}\,([\![t_i]\!]\,[\![s]\!])([\![s]\!]\,[\![t_i]\!]) \\ & \quad \text{by definition of } \mathsf{abs}_P^{A_i{}^{\oplus}} \\ = & [\![(t_i \bullet^{+} s) \bowtie_P (t_i \bullet^{-} s)]\!] \\ = & [\![t_i \bowtie_P s]\!] \end{array}$$

- **Absurdity Injection-Pair** (absInjPair): symmetric to the previous case.

- **Absurdity Negation** (absNeg): Let $\Gamma \vdash t : A^{\ominus}$ and $\Gamma \vdash s : A^{\oplus}$. Then:

$$\begin{array}{rl} & [\![(\nu^+ t) \bowtie_P (\nu^- s)]\!] \\ = & \mathsf{abs}_P^{(\neg A)^+} (\lambda x^{\mathbf{1}}. [\![t]\!])(\lambda y^{\mathbf{1}}. [\![s]\!]) \\ & \quad \text{where } x \notin \mathsf{fv}(t),\ y \notin \mathsf{fv}(s) \\ \to^+ & \mathsf{abs}_P^{A^{\ominus}} ((\lambda x^{\mathbf{1}}. [\![t]\!])\,\star)((\lambda y^{\mathbf{1}}. [\![s]\!])\,\star) \\ & \quad \text{by definition of } \mathsf{abs}_P^{(\neg A)^+} \\ \to^+ & \mathsf{abs}_P^{A^{\ominus}}\ [\![t]\!]\ [\![s]\!] \\ \to^+ & \mathsf{abs}_P^{A^-}\ ([\![t]\!]\ [\![s]\!])\ ([\![s]\!]\ [\![t]\!]) \\ & \quad \text{by definition of } \mathsf{abs}_P^{A^{\ominus}} \\ = & [\![(t \bullet^- s) \bowtie_P (s \bullet^+ t)]\!] \\ = & [\![t \bowtie_P s]\!] \end{array}$$

□

Theorem 40. *The λ^{PRK}-calculus is strongly normalizing.*

Proof. An easy consequence of Lem. 39 given that the extended System F is strongly normalizing (Coro. 35). □

2.4 Canonical proofs

In the previous sections we have shown that the λ^{PRK}-calculus enjoys subject reduction and strong normalization. This implies that each typable term t reduces to a normal form t' of the same type. In this subsection, these results are refined to prove a *canonicity* theorem, stating that each closed, typable term t reduces to a *canonical* term t' of the same type. For example, canonical terms of type $(A \vee B)^+$ are of the form $\mathsf{in}_i^+(t)$. From the logical point of view, this means that given a strong proof of $(A \vee B)$, in a context without assumptions, one can always recover either a classical proof of A or a classical proof of B. This shows that PRK has a form of disjunctive property.

First we provide an inductive characterization of the set of **normal forms** of λ^{PRK}.

Definition 41 (Normal terms)**.** The sets of *normal terms* $(N, \ldots)$ and *neutral terms* $(S, \ldots)$ are defined mutually inductively by:

$$\begin{array}{rclclcl} N & ::= & S & \mid & \langle N, N\rangle^{\pm} & \mid & \mathsf{in}_i^{\pm}(N) \\ & \mid & \nu^{\pm} N & \mid & \mathsf{IC}^{\pm}_{x:P}. N & & \\ \\ S & ::= & x & \mid & \pi_i^{\pm}(S) & \mid & \delta^{\pm} S\,[_x.N][_x.N] \\ & \mid & \mu^{\pm} S & \mid & S \bullet^{\pm} N & & \\ & \mid & S \bowtie_P N & \mid & N \bowtie_P S & & \end{array}$$

Proposition 42. *A term is normal if and only if it does not reduce in λ^{PRK}.*

Proof. ($\Rightarrow$) Let t be a normal term, and let us check that it is a $\to$-normal form. We proceed by structural induction on t.

The cases corresponding to introduction rules are straightforward by IH. For example, if $t = \langle N_1, N_2\rangle^{\pm}$, then by IH N_1 and N_2 have no $\to$-redexes. Moreover, there are no rules involving a pair $\langle -, -\rangle^{\pm}$ at the root, so $\langle N_1, N_2\rangle^{\pm}$ is in $\to$-normal form.

The cases corresponding to elimination rules and the absurdity rule are also straightforward by IH, observing that there cannot be a redex at the root. For example, if $t = \pi_i^{\pm}(S)$, then by IH S has no $\to$-redexes. Moreover, the only rule involving a projection $\pi_i^{\pm}(-)$ at the root is proj, which would require that $S = \langle t_1, t_2 \rangle^{\pm}$. But this is impossible —as can be checked by exhaustive case analysis on S—, so t is in $\to$-normal form.

($\Leftarrow$) Let t be a $\to$-normal form, let us check that it is a normal term. We proceed by induction on the structure of the term t:

- **Variable** (x): it is a neutral term.
- **Absurdity** ($t \bowtie_P s$): by IH, t and s are normal terms. If either t or s is a neutral term, we are done. We are left to analyze the case in which they are not neutral terms, *i.e.* both t and s are built using introduction rules. Note that the types of t and s are A^+ and A^- respectively. We proceed by case analysis on the form of the proposition A. There are four cases:
  1. **Propositional variable** ($A = \alpha$): This case is impossible, since t only may be of one of the following forms: $\langle N, N \rangle^+$, $\mathsf{in}_i^+(N)$, $\mathsf{IC}^+_{x:P}.\,N$, or $\nu^+ N$, none of which are of type α^+.
  2. **Conjunction** ($A = (B \wedge C)$): Then t is of the form $\langle t_1, t_2 \rangle^+$ and s is of the form $\mathsf{in}_i^-(s')$ for some $i \in \{1, 2\}$, so the rule $\mathsf{absPairInj}$ may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
  3. **Disjunction** ($A = B \vee C$): Then t is of the form $\mathsf{in}_i^+(s')$ for some $i \in \{1, 2\}$ and s is of the form $\langle t_1, t_2 \rangle^-$, so the rule $\mathsf{absInjPair}$ may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
  4. **Negation** ($A = \neg A$): Then t is of the form $\nu^+ t'$ and s is of the form $\nu^- s'$, so the rule absNeg may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
- **Pair** ($\langle t, s \rangle^{\pm}$): by IH, t and s are normal terms, so $\langle t, s \rangle^{\pm}$ is also a normal term.
- **Projection** ($\pi_i^{\pm}(t)$): by IH, t is a normal term. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t may be either of the form $(A \wedge B)^+$ or of the form $(A \vee B)^-$, we have that t is of the form $\langle s, u \rangle^{\pm}$. Then the rule proj may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
- **Injection** ($\mathsf{in}_i^{\pm}(t)$): by IH, t is a normal term, so $\mathsf{in}_i^{\pm}(t)$ is also normal.
- **Case** ($\delta^{\pm} t\,[_x.s][_x.u]$): by IH t, s and u are normal terms. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t may be either of the form $(A \vee B)^+$ or of the form $(A \wedge B)^-$, we have that t is of the form $\mathsf{in}_i^{\pm}(t')$ for some $i \in \{1, 2\}$. Then the rule case may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
- **Negation introduction** ($\nu^{\pm} t$): by IH, t is a normal term. Then $\nu^{\pm} t$ is also normal.
- **Negation elimination** ($\mu^{\pm} t$): by IH, t is a normal term. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t is of the form $(\neg A)^{\pm}$, then t is of the form $\nu^{\pm} t'$. Then the rule neg may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.

- **Classical introduction** ($\mathsf{IC}^{\pm}_{x:P}.\,t$): by IH, t is a normal term, so $\mathsf{IC}^{\pm}_{x:P}.\,t$ is also normal.

- **Classical elimination** ($t \bullet^{\pm} s$): by IH, t and s are normal terms. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t may be either of the form $A^{\oplus}$ or of the form $A^{\ominus}$, we have that t is of the form $\mathsf{IC}^{\pm}_{x}.\,t'$. Then the rule beta may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.

□

In order to state a canonicity theorem succintly, we introduce some nomenclature. A term is *canonical* if it has any of the following shapes:

$$\langle t_1, t_2\rangle^{\pm} \quad \mathsf{in}^{\pm}_i(t) \quad \nu^{\pm}t \quad \mathsf{IC}^{\pm}_{x}.\,t$$

i.e. it's normal, but not neutral, at the root.

A typing context is *classical* if all the assumptions are classical, *i.e.* of the form $A^{\oplus}$ or $A^{\ominus}$. A *case-context* is a context of the form $\mathtt{K} ::= \Box \mid \delta^{\pm}\mathtt{K}\,[_x.t][_y.s]$. An *eliminative context* is a context of the form $\mathtt{E} ::= \Box \mid \pi^{\pm}_i(\mathtt{E}) \mid \mu^{\pm}\mathtt{E} \mid \mathtt{K}\langle\mathtt{E}\rangle$. Note that $\Box \bullet^{\pm} t$ is not eliminative and that all case-contexts are eliminative. An *explosion* is a term of the form $t \bowtie_P s$ or of the form $t \bullet^{\pm} s$. A term is *closed* if it has no free variables. A term is *open* if it not closed, *i.e.* it has at least one free variable.

The following theorem has three parts; the first one provides guarantees for *closed* terms, whereas the two other ones provide weaker guarantees for terms typable under an arbitrary classical context.

Theorem 43 (Canonicity)**.**

1. *Let $\vdash t : P$ where t is a normal form. Then t is canonical.*

2. *Let $\Gamma \vdash t : A^{\pm}$ where Γ is classical and t is a normal form. Then either t is canonical or t is of the form $\mathtt{K}\langle t'\rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion.*

3. *Let $\Gamma \vdash t : A^{\oplus}$ or $\Gamma \vdash t : A^{\ominus}$, where Γ is classical and t is a normal form. Then either $t = \mathsf{IC}^{\pm}_{x}.\,t'$ or $t = \mathtt{E}\langle t''\rangle$, where $\mathtt{E}$ is an eliminative context and t'' is a variable or an open explosion.*

Proof.

1. Let $\vdash t : P$ where t is a normal form. Note, by induction on the formation rules for neutral terms (Def. 41) that a neutral term must have at least one free variable. But t is typed in the empty typing context, so it must be closed. Hence t is not a neutral term, so by Prop. 42, it must be canonical.

2. Let $\Gamma \vdash t : P$ where Γ is classical and t is a normal form. By Prop. 42 either t is canonical or it is a neutral term. If t is canonical we are done. If t is a neutral term it suffices to show the following claim, namely that if $\Gamma \vdash t : B^{\pm}$ is a derivable judgment such that Γ is classical and t is a neutral term, then t is of the form $t = \mathtt{K}\langle t'\rangle$, where $\mathtt{K}$ is a case-context and t' is an open explosion. We proceed by induction on the formation rules for neutral terms (Def. 41):

- **Variable** ($t = x.$): this case is impossible, given that Γ is assumed to be classical, so $\Gamma \vdash x : P$ where P must be of the form $C^{\oplus}$ or $C^{\ominus}$, hence P cannot be of the form $B^{\pm}$.
- **Projection** ($\pi_i^{\pm}(S)$): this case is impossible, as $\Gamma \vdash \pi_i^{\pm}(S) : P$ where P must be of the form $C^{\oplus}$ or $C^{\ominus}$, hence P cannot be of the form $B^{\pm}$.
- **Case** ($\delta^{\pm} S\,[_x.N_1][_x.N_2]$): by inversion of the typing rules we have that either $\Gamma \vdash S : (A \vee B)^{+}$ or $\Gamma \vdash S : (A \wedge B)^{-}$. In both cases we may apply the IH to conclude that S is of the form $S = \mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \delta^{\pm}(\mathtt{K}\langle t' \rangle)\,[_x.N_1][_x.N_2]$ where now $\delta^{\pm}(\mathtt{K})\,[_x.N_1][_x.N_2]$ is a case-context.
- **Negation elimination** ($\mu^{\pm} S$): this case is impossible, as $\Gamma \vdash \mu^{\pm} S : P$ where P must be of the form $C^{\oplus}$ or $C^{\ominus}$, hence P cannot be of the form $B^{\pm}$.
- **Classical elimination** ($S \bullet^{\pm} N$): then t is an explosion under the empty case-context. Moreover, S must have at least one free variable so t is indeed an open explosion.
- **Absurdity** ($S \bowtie N$ or $N \bowtie S$): then t is an explosion under the empty case-context. Moreover, S must have at least one free variable so t is indeed an open explosion.

3. Let $\Gamma \vdash t : A^{\oplus}$ or $\Gamma \vdash t : A^{\ominus}$, where Γ is classical and t is a normal form. By Prop. 42 either t is canonical or it is a neutral term. If t is canonical, then by the constraints on its type it must be of the form $t = \mathsf{IC}_x^{\pm}.\,t'$, so we are done. If t is neutral, it suffices to show the following claim namely that if $\Gamma \vdash t : P$ is a derivable judgment, with P classical, such that Γ is classical and t is a neutral term, then t is of the form $t = \mathtt{E}\langle t' \rangle$, where $\mathtt{E}$ is an eliminative context and t' is a variable or an open explosion. We proceed by induction on the formation rules for neutral terms (Def. 41):

   - **Variable** ($t = x$): immediate, as t is a variable under the empty eliminative context.
   - **Projection** ($\pi_i^{\pm}(S)$): by inversion of the typing rules, we have that either $\Gamma \vdash S : (A \wedge B)^{+}$ or $\Gamma \vdash S : (A \vee B)^{-}$. In both cases we may apply the second item of this lemma to conclude that S is of the form $S = \mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \pi_i^{\pm}(\mathtt{K}\langle t' \rangle)$, where now $\pi_i^{\pm}(\mathtt{K})$ is an eliminative context.
   - **Case** ($\delta^{\pm} S\,[_x.N_1][_x.N_2]$): by inversion of the typing rules, we have that either $\Gamma \vdash S : (A \vee B)^{+}$ or $\Gamma \vdash S : (A \wedge B)^{-}$. In both cases we may apply the second item of this lemma to conclude that S is of the form $S = \mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is an eliminative context and t' is an open explosion. Therefore $t = \delta^{\pm}(\mathtt{K}\langle t' \rangle)\,[_x.N_1][_x.N_2]$, where now $\delta^{\pm}(\mathtt{K})\,[_x.N_1][_x.N_2]$ is an eliminative context.
   - **Negation elimination** ($\mu^{\pm} S$): by inversion of the typing rules, we have that $\Gamma \vdash S : (\neg A)^{\pm}$. By the second item of this lemma, S is of the form $S = \mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \mu^{\pm}\mathtt{K}\langle t' \rangle$, where now $\mu^{\pm}\mathtt{K}$ is an eliminative context.
   - **Classical elimination** ($S \bullet^{\pm} N$): then t is an explosion under the empty eliminative context. Moreover, S must have at least one free variable so t is indeed an open explosion.

- **Absurdity** ($S \bowtie N$ or $N \bowtie S$): then t is an explosion under the empty eliminative context. Moreover, S must have at least one free variable so t is indeed an open explosion.

□

3 Extension: classical extensionality

To conclude the syntactic study of λ^{PRK}, we discuss that an extensionality rule, similar to η-reduction in the λ-calculus, may be incorporated to λ^{PRK}, obtaining a calculus $\lambda^{\text{PRK}}_{\eta}$ that still enjoys subject reduction, strong normalization, and confluence.

Definition 44. The $\lambda^{\text{PRK}}_{\eta}$-calculus is defined by extending the λ^{PRK} calculus with the following reduction rule:

$$\mathsf{IC}^{\pm}_{x}.\,(t \bullet^{\pm} x) \xrightarrow{\mathsf{eta}} t \quad \text{if } x \notin \mathsf{fv}(t)$$

It is straightforward to show that $\lambda^{\text{PRK}}_{\eta}$ enjoys subject reduction, extending the proof of Prop. 31 with an easy case for the eta rule. Furthermore:

Lemma 45 (Local confluence)**.** *The $\lambda^{\text{PRK}}_{\eta}$-calculus has the weak Church–Rosser property.*

Proof. Let $t_0 \to t_1$ and $t_0 \to t_2$, and let us show that the diagram can be closed, *i.e.* that there is a term t_3 such that $t_1 \to^* t_3$ and $t_2 \to^* t_3$. The proof is by induction on t_0 and by case analysis on the relative positions of the steps $t_0 \to t_1$ and $t_0 \to t_2$. Most cases are straightforward by resorting to the IH. We study only the interesting cases, when the patterns of the redexes overlap. There are two such cases:

1. $\mathsf{beta}/\mathsf{eta}$: Let $x \notin \mathsf{fv}(t)$. The overlap involves a step $(\mathsf{IC}^{\pm}_{x}.\,t \bullet^{\pm} x) \bullet^{\pm} s \xrightarrow{\mathsf{beta}} t \bullet^{\pm} s$ and a step $(\mathsf{IC}^{\pm}_{x}.\,t \bullet^{\pm} x) \bullet^{\pm} s \xrightarrow{\mathsf{eta}} t \bullet^{\pm} s$, so the diagram is trivially closed in zero rewriting steps.

2. $\mathsf{eta}/\mathsf{beta}$: Let $x \notin \mathsf{fv}(t)$. The overlap involves a step $\mathsf{IC}^{\pm}_{x}.\,(\mathsf{IC}^{\pm}_{y}.\,t) \bullet^{\pm} x \xrightarrow{\mathsf{eta}} \mathsf{IC}^{\pm}_{y}.\,t$ and a step $\mathsf{IC}^{\pm}_{x}.\,(\mathsf{IC}^{\pm}_{y}.\,t) \bullet^{\pm} x \xrightarrow{\mathsf{beta}} \mathsf{IC}^{\pm}_{x}.\,t[y := x]$. Note that the targets of the steps are α-equivalent, so the diagram is trivially closed in zero rewriting steps.

□

Lemma 46 (Properties of reduction in $\lambda^{\text{PRK}}_{\eta}$)**.**

1. Reduction does not create free variables. *If $t \to t'$ then* $\mathsf{fv}(t) \supseteq \mathsf{fv}(t')$.

2. Substitution (I). *Let $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. If $t \to t'$ then $t[x := s] \to t'[x := s]$.*

3. Substitution (II). *Let $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. If $s \to s'$ then $t[x := s] \to^* t[x := s']$.*

4. Substitution (III). *Let $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. If $t \to^* t'$ and $s \to^* s'$ then $t[x := s] \to^* t'[x := s']$.*

Proof. Items 1., 2., and 3. are by induction on t. Item 4. is by induction on the sum of the lengths of the sequences $t \to^* t'$ and $s \to^* s'$, resorting to the two previous items. □

Lemma 47 (Postponement of eta steps)**.** *Let* $t \xrightarrow{\mathsf{eta}} s \xrightarrow{\mathsf{r}} u$ *where* r *is a rewriting rule other than* eta*. Then there exists a term* s' *such that* $t \xrightarrow{\mathsf{r}}{}^{+} s' \xrightarrow{\mathsf{eta}}{}^{*} u$*.*

Proof. By induction on t. If the eta step and the r step are not reduction steps at the root, it is immediate to conclude, resorting to the IH when appropriate.

If the eta step is at the root, then the first step is of the form $t = \mathsf{IC}^{\pm}_{x}.\,(s \bullet^{\pm} x) \xrightarrow{\mathsf{eta}} s$, where $x \notin \mathsf{fv}(s)$. Taking $s' := \mathsf{IC}^{\pm}_{x}.\,(u \bullet^{\pm} x)$ we have that $t = \mathsf{IC}^{\pm}_{x}.\,(s \bullet^{\pm} x) \xrightarrow{\mathsf{r}} \mathsf{IC}^{\pm}_{x}.\,(u \bullet^{\pm} x) \xrightarrow{\mathsf{eta}} u$, so we are done. For the last reduction step, we use the fact that reduction does not create free variables (Lem. 46).

Otherwise, we have that the eta step is *not* at the root and the r step is at the root. Then we proceed by case analysis, depending on the kind of rule applied. We only study the positive cases (the negative cases are symmetric):

- **Projection** (proj): then we have that $t \xrightarrow{\mathsf{eta}} s = \pi^{+}_{i}(\langle s_1, s_2\rangle^{+}) \xrightarrow{\mathsf{proj}} s_i$. Recall that the eta step is not at the root of t. Moreover, it cannot be the case that $t = \pi^{+}_{i}(t')$ and the eta step is at the root of t', because the type of t' must be of the form $(A \wedge B)^{+}$ but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^{\pm}_{-}.\,-$, whose type is classical. This means that t must be of the form $\pi^{+}_{i}(\langle t_1, t_2\rangle^{+})$ and that the eta step is either internal to t_1 or internal to t_2, which implies that $t_1 \xrightarrow{\mathsf{eta}}{}^{*} s_1$ and $t_2 \xrightarrow{\mathsf{eta}}{}^{*} s_2$. Taking $s' := t_i$ we have that $t = \pi^{+}_{i}(\langle t_1, t_2\rangle^{+}) \xrightarrow{\mathsf{proj}} t_i \xrightarrow{\mathsf{eta}}{}^{*} s_i$, as required.

- **Case** (case): then we have that $t \xrightarrow{\mathsf{eta}} s = \delta^{+}\mathsf{in}^{+}_{i}(s_0)\,[_{y}.s_1][_{y}.s_2] \xrightarrow{\mathsf{case}} s_i[y := s_0]$. Recall that the eta step is not at the root of t. Moreover, it cannot be the case that $t = \delta^{+}t'\,[_{y}.s_1][_{y}.s_2]$ and the eta step is at the root of t', because the type of t' must be of the form $(A \vee B)^{+}$, but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^{\pm}_{-}.\,-$, whose type is classical. This means that t must be of the form $\delta^{+}\mathsf{in}^{+}_{i}(t_0)\,[_{y}.t_1][_{y}.t_2]$ and that the eta-step is either internal to t_0, or internal to t_1, or internal to t_2, which implies that $t_0 \xrightarrow{\mathsf{eta}}{}^{*} s_0$ and $t_1 \xrightarrow{\mathsf{eta}}{}^{*} s_1$ and $t_2 \xrightarrow{\mathsf{eta}}{}^{*} s_2$. Taking $s' := t_i[y := t_0]$ we have that $t = \delta^{+}\mathsf{in}^{+}_{i}(t_0)\,[_{y}.t_1][_{y}.t_2] \xrightarrow{\mathsf{case}} t_i[y := t_0] \xrightarrow{\mathsf{eta}}{}^{*} s_i[y := s_0]$ resorting to Lem. 46 for the last step.

- **Negation** (neg): then we have that $t \xrightarrow{\mathsf{eta}} \mu^{+}(\nu^{+}s_1) \xrightarrow{\mathsf{neg}} s_1$. Recall that the eta-reduction step is not at the root of t. Moreover, it cannot be the case that $t = \mu^{+}t'$ and the eta-reduction step is at the root of t', because the type of t' must be of the form $(\neg A)^{+}$ but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^{\pm}_{-}.\,-$, whose type is classical. This means that t must be of the form $\mu^{+}(\nu^{+}t_1)$ and that the eta step is internal to t_1, *i.e.* $t_1 \xrightarrow{\mathsf{eta}} s_1$. Then taking $s' := t_1$ we have that $t = \mu^{+}(\nu^{+}t_1) \xrightarrow{\mathsf{neg}} t_1 \xrightarrow{\mathsf{eta}} s_1$ as required.

- **Beta** (beta): then we have that $t \xrightarrow{\mathsf{eta}} (\mathsf{IC}^{+}_{y}.\,s_1) \bullet^{+} s_2 \xrightarrow{\mathsf{beta}} s_1[y := s_2]$. Recall that the eta step is not at the root of t. There are three cases, depending on the position of the eta-step:

1. *Immediately to the left of the application.* That is, $t = t' \bullet^+ s_2$ and the eta step is at the root of t', *i.e.* $t' \xrightarrow{\mathsf{eta}} \mathsf{IC}^+_y.\, s_1$ is a reduction step at the root. Then $t' = \mathsf{IC}^+_x.\, ((\mathsf{IC}^+_y.\, s_1) \bullet^+ x)$, and $x \notin \mathsf{fv}(s_1)$. Hence taking $s' := s_1[y := s_2]$ we have that
$$\begin{array}{lll} & t = (\mathsf{IC}^+_x.\, ((\mathsf{IC}^+_y.\, s_1) \bullet^+ x)) \bullet^+ s_2 & \\ \xrightarrow{\mathsf{beta}} & (\mathsf{IC}^+_y.\, s_1) \bullet^+ s_2 & x \notin \mathsf{fv}(s_1) \\ \xrightarrow{\mathsf{beta}} & s_1[y := s_2] & \end{array}$$
using two beta steps and no eta steps.
2. *Inside the abstraction.* That is, $t = (\mathsf{IC}^+_y.\, t_1) \bullet^+ s_2$ with $t_1 \xrightarrow{\mathsf{eta}} s_1$. Then taking $s' := t_1[y := s_2]$ we have that $t = (\mathsf{IC}^+_y.\, t_1) \bullet^+ s_2 \xrightarrow{\mathsf{beta}} t_1[y := s_2] \xrightarrow{\mathsf{eta}} s_1[y := s_2]$ resorting to Lem. 46 for the last step.
3. *To the right of the application.* That is, $t = (\mathsf{IC}^+_y.\, s_1) \bullet^+ t_2$ with $t_2 \xrightarrow{\mathsf{eta}} s_2$. Then taking $s' := s_1[y := t_2]$ we have that $t = (\mathsf{IC}^+_y.\, s_1) \bullet^+ t_2 \xrightarrow{\mathsf{beta}} s_1[y := t_2] \xrightarrow{\mathsf{eta}}{}^* s_1[y := s_2]$ resorting to Lem. 46 for the last step.

- **Absurdity Pair-Injection** ($\mathsf{absPairInj}$): then we have that $t \xrightarrow{\mathsf{eta}} \langle s_1, s_2\rangle^+ \bowtie \mathsf{in}^-_i(s_3) \xrightarrow{\mathsf{absPairInj}} s_i \bowtie s_3$. Recall that the eta step is not at the root of t. Moreover, it cannot be the case that $t = t' \bowtie \mathsf{in}^-_i(s_3)$ and the eta step is at the root of t', because the type of t' must be of the form $(A \wedge B)^+$, but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^\pm_-.\, -$, whose type is classical. For similar reasons, it cannot be the case that $t = \langle s_1, s_2\rangle^+ \bowtie t'$ with the eta step is at the root of t'. This means that t must be of the form $\langle t_1, t_2\rangle^+ \bowtie \mathsf{in}^-_i(t_3)$ and that the eta step is either internal to t_1, internal to t_2, or internal to t_3. This implies that $t_1 \xrightarrow{\mathsf{eta}}{}^* s_1$ and $t_2 \xrightarrow{\mathsf{eta}}{}^* s_2$ and $t_3 \xrightarrow{\mathsf{eta}}{}^* s_3$. Taking $s' := t_i \bowtie t_3$ we have that $t = \langle t_1, t_2\rangle^+ \bowtie \mathsf{in}^-_i(t_3) \xrightarrow{\mathsf{absPairInj}} t_i \bowtie t_3 = (t_i \bullet^+ t_3) \bowtie (t_3 \bullet^- t_i) \xrightarrow{\mathsf{eta}}{}^* (s_i \bullet^+ s_3) \bowtie (s_3 \bullet^- s_i) = s_i \bowtie s_3$.
- **Absurdity Injection-Pair** ($\mathsf{absInjPair}$): Symmetric to the previous case.
- **Absurdity Negation** (absNeg): then we have that $t \xrightarrow{\mathsf{eta}} (\nu^+ s_1) \bowtie (\nu^- s_2) \xrightarrow{\mathsf{absNeg}} s_1 \bowtie s_2$. Recall that the eta step is not at the root of t. Moreover, it cannot be the case that $t = t' \bowtie (\nu^- s_2)$ and the eta step is at the root of t', because the type of t' must be of the form $(\neg A)^+$, but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^\pm_-.\, -$, whose type is classical. For similar reasons, it cannot be the case that $t = \nu^+ s_1 \bowtie t'$ with the eta step is at the root of t'. This means that t must be of the form $(\nu^+ t_1) \bowtie (\nu^- t_2)$ and that the eta step is either internal to t_1 or internal to t_2. This implies that $t_1 \xrightarrow{\mathsf{eta}}{}^* s_1$ and $t_2 \xrightarrow{\mathsf{eta}}{}^* s_2$. Taking $s' := t_1 \bowtie t_2$ we have that $t = (\nu^+ t_1) \bowtie (\nu^- t_2) \xrightarrow{\mathsf{absNeg}} t_1 \bowtie t_2 = (t_2 \bullet^+ t_1) \bowtie (t_1 \bullet^- t_2) \xrightarrow{\mathsf{eta}}{}^* (s_2 \bullet^+ s_1) \bowtie (s_1 \bullet^- s_2) = s_1 \bowtie s_2$.

□

Theorem 48. *The $\lambda^{\mathrm{PRK}}_\eta$-calculus is strongly normalizing and confluent.*

Proof. Strong normalization follows from postponement of the eta rule (Lem. 47) and strong normalization of the calculus without eta (Thm. 40) by the usual rewriting techniques.

More precisely, let us write $\xrightarrow{\neg\mathsf{eta}}$ for reduction not using eta, that is, $\xrightarrow{\neg\mathsf{eta}} \stackrel{\text{def}}{=} (\to \setminus \xrightarrow{\mathsf{eta}})$. Suppose there is an infinite reduction sequence $t_1 \to t_2 \to t_3 \ldots$ in $\lambda^{\text{PRK}}_{\eta}$. Let $t_1 \xrightarrow{\neg\mathsf{eta}}{}^* \; t_i$ be the longest prefix of the sequence whose steps are not eta steps. This prefix cannot be infinite given that λ^{PRK} is strongly normalizing.

Let $t_i \xrightarrow{\mathsf{eta}}{}^* \; t_{i+n}$ be the longest sequence of eta steps starting on t_i. This sequence cannot be infinite given that an eta step strictly decreases the size of the term. Now there must be a step $t_{i+n} \xrightarrow{\neg\mathsf{eta}} t_{i+n+1}$. Applying the postponement lemma (Lem. 47) n times, we obtain a sequence of the form $t_1 \xrightarrow{\neg\mathsf{eta}}{}^* \; t_i \xrightarrow{\neg\mathsf{eta}} t'_{i+1} \ldots$. By repeatedly applying this argument, we may build an infinite sequence of $\xrightarrow{\neg\mathsf{eta}}$ steps, contradicting the fact that λ^{PRK} is strongly normalizing.

Confluence of $\lambda^{\text{PRK}}_{\eta}$ follows from the fact that it is strongly normalizing and locally confluent (Lem. 45), resorting to Newman's Lemma [13, Theorem 1.2.1]. □

5. RELATIONSHIP WITH CLASSICAL LOGIC

Intuitionistic logic *refines* classical logic: each intuitionistically valid formula A is also classically valid, but there may be many classically equivalent "readings" of a formula which are not intuitionistically equivalent, such as $\neg(A \wedge \neg B)$ and $\neg A \vee B$.

System PRK refines classical logic in a similar sense. For example, the classical sequent $\alpha \vdash \alpha$ may be "read" in PRK in various different ways, such as $\alpha^+ \vdash \alpha^\oplus$ and $\alpha^\oplus \vdash \alpha^+$, of which the former holds but the latter does not. In this section we show that PRK is *conservative* (Prop. 49) with respect to classical logic, and that classical logic may be *embedded* (Thm. 50) in PRK. We also describe the computational behavior of the terms resulting from this embedding (Lem. 51).

As a note, we base our study on NK, a Natural Deduction formalization of classical logic, shown in the Appendix 1.

First, we claim that PRK is a **conservative extension** of classical logic, *i.e.* if $A_1^\oplus, \ldots, A_n^\oplus \vdash B^\oplus$ holds in PRK then the sequent $A_1, \ldots, A_n \vdash B$ holds in classical logic. More in general:

Proposition 49. *Define $c(P)$ as follows:*

$$c(A^\oplus) \stackrel{\text{def}}{=} A \quad c(A^\ominus) \stackrel{\text{def}}{=} \neg A$$
$$c(A^+) \stackrel{\text{def}}{=} A \quad c(A^-) \stackrel{\text{def}}{=} \neg A$$

If the sequent $P_1, \ldots, P_n \vdash Q$ holds in PRK *then the sequent $c(P_1), \ldots, c(P_n) \vdash c(Q)$ holds in classical propositional logic.*

Proof. By induction on the derivation of the judgment, observing that all the inference rules in PRK are mapped to classically valid inferences. For example, for the E$\wedge^-$ rule, note that if $\Gamma \vdash \neg(A \wedge B)$ and $\Gamma, \neg A \vdash C$ and $\Gamma, \neg B \vdash C$ hold in classical propositional logic then $\Gamma \vdash C$ also holds. □

Second, we claim that classical logic may be **embedded** in PRK, that is:

Theorem 50. *If $A_1, \ldots, A_n \vdash B$ holds in classical logic then $A_1^\oplus, \ldots, A_n^\oplus \vdash B^\oplus$ holds in* PRK.

Proof. The proof is by induction on the proof of the sequent $A_1, \ldots, A_n \vdash B$ in Gentzen's system of natural deduction for classical logic NK, including introduction and elimination rules for conjunction, disjunction, and negation (encoding falsity as the pure proposition $\bot \stackrel{\text{def}}{=} (\alpha_0 \wedge \neg\alpha_0)$ for some fixed propositional variable α_0), the explosion principle, and the law of excluded middle. We build the corresponding proof terms in λ^{PRK}:

- **Conjunction introduction**: Let $\Gamma \vdash t : A^\oplus$ and $\Gamma \vdash s : B^\oplus$. Then $\Gamma \vdash \langle t, s\rangle^{\mathcal{C}} : (A \wedge B)^\oplus$ where:
$$\langle t, s\rangle^{\mathcal{C}} \stackrel{\text{def}}{=} \mathsf{IC}^+_{(\_:(A \wedge B)^\ominus)}.\, \langle t, s\rangle^+$$
- **Conjunction elimination**: Let $\Gamma \vdash t : (A_1 \wedge A_2)^\oplus$. Then $\Gamma \vdash \pi_i^{\mathcal{C}}(t) : A_i{}^\oplus$ where:
$$\pi_i^{\mathcal{C}}(t) \stackrel{\text{def}}{=} \mathsf{IC}^+_{(x:A_i{}^\ominus)}.\, \pi_i^+(t \bullet^+ \mathsf{IC}^-_{(\_:(A_1 \wedge A_2)^\oplus)}.\, \mathsf{in}_i^-(x)) \bullet^+ x$$

- **Disjunction introduction**: Let $\Gamma \vdash t : A_i{}^{\oplus}$. Then $\Gamma \vdash \mathsf{in}_i^{\mathcal{C}}(t) : (A_1 \vee A_2)^{\oplus}$ where:

$$\mathsf{in}_i^{\mathcal{C}}(t) \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{(\_:(A_1 \vee A_2)^{\ominus})}.\, \mathsf{in}_i^{+}(t)$$

- **Disjunction elimination**: Let $\Gamma \vdash t : (A \vee B)^{\oplus}$ and $\Gamma, x : A^{\oplus} \vdash s : C^{\oplus}$ and $\Gamma, x : B^{\oplus} \vdash u : C^{\oplus}$. Then $\Gamma \vdash \delta^{\mathcal{C}} t\, [_{(x:A^{\oplus})}.s][_{(x:B^{\oplus})}.u] : C^{\oplus}$, where:

$$\begin{array}{lll} \delta^{\mathcal{C}} t\, [_{(x:A^{\oplus})}.s][_{(x:B^{\oplus})}.u] \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{(y:C^{\ominus})}. & \delta^{+} & (t \bullet^{+} \mathsf{IC}^{-}_{(\_:(A \vee B)^{\oplus})}.\, \langle \updownarrow_x^y (s), \updownarrow_x^y (u)\rangle^{-}) \\ & & [_{(x:A^{\oplus})}.s \bullet^{+} y] \\ & & [_{(x:B^{\oplus})}.u \bullet^{+} y] \end{array}$$

Recall that $\updownarrow_x^y (t)$ stands for the witness of contraposition (Lem. 27).

- **Negation introduction**: By Lem. 27 we have that $\Gamma \vdash \pitchfork^{-}_{\alpha_0} : (\alpha_0 \wedge \neg\alpha_0)^{\ominus}$, that is $\Gamma \vdash \pitchfork^{-}_{\alpha_0} : \bot^{\ominus}$. Moreover, suppose that $\Gamma, x : A^{\oplus} \vdash t : \bot^{\oplus}$. Then $\Gamma \vdash \Lambda^{\mathcal{C}}_{(x:A^{\oplus})}.\, t : (\neg A)^{\oplus}$, where:

$$\Lambda^{\mathcal{C}}_{(x:A^{\oplus})}.\, t \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{(\_:(\neg A)^{\ominus})}.\, \nu^{+} \mathsf{IC}^{-}_{(x:A^{\oplus})}.\, (t \bowtie_{A^{-}} \pitchfork^{-}_{\alpha_0})$$

- **Negation elimination**: Let $\Gamma \vdash t : (\neg A)^{\oplus}$ and $\Gamma \vdash s : A^{\oplus}$. Then $\Gamma \vdash t \#^{\mathcal{C}} s : \bot^{\oplus}$, where:

$$t \#^{\mathcal{C}} s \stackrel{\text{def}}{=} t \bowtie_{\bot^{\oplus}} \mathsf{IC}^{-}_{(\_:(\neg A)^{\oplus})}.\, \nu^{-} s$$

- **Explosion**: Let $\Gamma \vdash t : \bot^{\oplus}$. Then $\Gamma \vdash (t \bowtie_{Q} \pitchfork^{-}_{\alpha_0}) : Q$.

- **Excluded middle**: It suffices to take $\pitchfork^{\mathcal{C}}_{A} \stackrel{\text{def}}{=} \pitchfork^{+}_{A}$. Then by Lem. 27, $\Gamma \vdash \pitchfork^{\mathcal{C}}_{A} : (A \vee \neg A)^{\oplus}$.

□

Finally, this embedding may be understood as providing a **computational interpretation** for classical logic. In fact, besides the introduction and elimination rules that have been proved above, implication may be defined as an abbreviation, $(A \Rightarrow B) \stackrel{\text{def}}{=} (\neg A \vee B)$, and witnesses for its introduction rule $\lambda^{\mathcal{C}}_{x:A}.\, t$ and its elimination rule $t \,@^{\mathcal{C}}\, s$ may be defined as follows. If $\Gamma, x : A^{\oplus} \vdash t : B^{\oplus}$ then $\Gamma \vdash \lambda^{\mathcal{C}}_{(x:A)}.\, t : (A \Rightarrow B)^{\oplus}$ where:

$$\begin{array}{rcl} \lambda^{\mathcal{C}}_{x}.\, t & \stackrel{\text{def}}{=} & \mathsf{IC}^{+}_{(y:(A \Rightarrow B)^{\ominus})}.\, \mathsf{in}_2^{+}(t[x := \mathbf{X}_y]) \\ \mathbf{X}_y & \stackrel{\text{def}}{=} & \mathsf{IC}^{+}_{(z:A^{\ominus})}.\, (\mu^{-}(\mathbf{X'}_{y,z} \bullet^{-} \mathsf{IC}^{+}_{(\_:(\neg A)^{\ominus})}.\, \nu^{+} z)) \bullet^{+} z \\ \mathbf{X'}_{y,z} & \stackrel{\text{def}}{=} & \pi_1^{+}(y \bullet^{-} \mathsf{IC}^{+}_{(\_:(A \Rightarrow B)^{\ominus})}.\, \mathsf{in}_1^{+}(\mathsf{IC}^{+}_{(\_:(\neg A)^{\ominus})}.\, \nu^{+} z)) \end{array}$$

If $\Gamma \vdash t : (A \Rightarrow B)^{\oplus}$ and $\Gamma \vdash s : A^{\oplus}$, then $\Gamma \vdash t \,@^{\mathcal{C}}\, s : B^{\oplus}$, where:

$$\begin{array}{lll} t \,@^{\mathcal{C}}\, s \stackrel{\text{def}}{=} & \mathsf{IC}^{+}_{(x:B^{\ominus})}. & \\ & \delta^{+} & (t \bullet^{+} \mathsf{IC}^{-}_{(\_:(A \Rightarrow B)^{\oplus})}.\, \langle (\mathsf{IC}^{-}_{(\_:(\neg A)^{\oplus})}.\, \nu^{-} s), x \rangle^{-}) \\ & & [_{(y:(\neg A)^{\oplus})}.s \bowtie_{B^{+}} \mu^{-}(y \bullet^{+} \mathsf{IC}^{-}_{(\_:(\neg A)^{\oplus})}.\, \nu^{-} x)] \\ & & [_{(z:B^{\oplus})}.z \bullet^{+} x] \end{array}$$

Lemma 51. *The following holds in* $\lambda_\eta^{\text{PRK}}$ *(with* eta *reduction):*

$$
\begin{array}{rcl}
\pi_i^{\mathcal{C}}(\langle t_1, t_2\rangle^{\mathcal{C}}) & \to^* & t_i \\
\delta^{\mathcal{C}}\mathsf{in}_i^{\mathcal{C}}(t)\,[_x.s_1][_x.s_2] & \to^* & s_i[x:=t] \\
(\lambda_x^{\mathcal{C}}.\,t)\,@^{\mathcal{C}}\,s & \to^* & t[x:=s] \\
\delta^{\mathcal{C}}\ \pitchfork_A^{\mathcal{C}}\ [_x.s_1][_x.s_2] & \to^* & \mathsf{IC}_y^+.\,(s_2[x:=s_1^*]\bullet^+ y)
\end{array}
$$

where $s_1^* := \mathsf{IC}_{\_}^+.\,\nu^+(\mathsf{IC}_x^-.\,s_1 \bowtie y)$.

Proof. By calculation. The last rule describes the behaviour of the law of excluded middle.

Simulation of conjunction:

$$
\begin{array}{rl}
& \pi_i^{\mathcal{C}}(\langle t_1, t_2\rangle^{\mathcal{C}}) \\
= & \mathsf{IC}_{x:A_i{}^\ominus}^+.\,\pi_i^+((\mathsf{IC}_{\_}^+.\,\langle t_1,t_2\rangle^+)\bullet^+ \mathsf{IC}_{\_}^-.\,\mathsf{in}_i^-(x))\bullet^+ x \\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}_{x:A_i{}^\ominus}^+.\,\pi_i^+(\langle t_1,t_2\rangle^+)\bullet^+ x \\
\xrightarrow{\mathsf{proj}} & \mathsf{IC}_{x:A_i{}^\ominus}^+.\,t_i\bullet^+ x \\
\xrightarrow{\mathsf{eta}} & t_i
\end{array}
$$

Simulation of disjunction:

$$
\begin{array}{rl}
& \delta^{\mathcal{C}}\mathsf{in}_i^{\mathcal{C}}(t)\,[_x.s_1][_x.s_2] \\
= & \mathsf{IC}_{(y:C^\ominus)}^+. \\
& \quad\delta^+\left(\begin{array}{l}\mathsf{IC}_{(\_:(A_1\vee A_2)^\ominus)}^+.\,\mathsf{in}_i^+(t) \\ \bullet^+\ \mathsf{IC}_{(\_:(A\vee B)^\oplus)}^-.\,\langle \updownarrow_x^y(s_1), \updownarrow_x^y(s_2)\rangle^-\end{array}\right) \\
& \quad[_{(x:A^\oplus)}.s_1\bullet^+ y] \\
& \quad[_{(x:B^\oplus)}.s_2\bullet^+ y] \\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}_{(y:C^\ominus)}^+.\,\delta^+\mathsf{in}_i^+(t)\,[_{(x:A^\oplus)}.s_1\bullet^+ y][_{(x:B^\oplus)}.s_2\bullet^+ y] \\
\xrightarrow{\mathsf{case}} & \mathsf{IC}_{(y:C^\ominus)}^+.\,s_i[x:=t]\bullet^+ y \\
\xrightarrow{\mathsf{eta}} & s_i[x:=t]
\end{array}
$$

Simulation of negation:

$$
\begin{array}{rl}
& (\Lambda_x^{\mathcal{C}}.\,t)\#^{\mathcal{C}}s \\
= & (\mathsf{IC}_{\_}^+.\,\nu^+\mathsf{IC}_x^-.\,(t\bowtie\pitchfork_{\alpha_0}^-))\bowtie_{\perp^\oplus}\mathsf{IC}_{\_}^-.\,\nu^- s \\
= & \quad((\mathsf{IC}_{\_}^+.\,\nu^+\mathsf{IC}_x^-.\,(t\bowtie\pitchfork_{\alpha_0}^-))\bullet^+\mathsf{IC}_{\_}^-.\,\nu^- s) \\
& \blacktriangleright\!\blacktriangleleft(\mathsf{IC}_{\_}^-.\,\nu^- s\bullet^-(\mathsf{IC}_{\_}^+.\,\nu^+\mathsf{IC}_x^-.\,(t\bowtie\pitchfork_{\alpha_0}^-))) \\
\xrightarrow{\mathsf{beta}}(2) & (\nu^+\mathsf{IC}_x^-.\,(t\bowtie\pitchfork_{\alpha_0}^-))\ \blacktriangleright\!\blacktriangleleft\ (\nu^- s) \\
\xrightarrow{\mathsf{absNeg}} & ((\mathsf{IC}_x^-.\,(t\bowtie\pitchfork_{\alpha_0}^-))\bullet^+ s)\ \blacktriangleright\!\blacktriangleleft\ (s\bullet^-(\mathsf{IC}_x^-.\,(t\bowtie\pitchfork_{\alpha_0}^-))) \\
\xrightarrow{\mathsf{beta}} & (t[x:=s]\bowtie\pitchfork_{\alpha_0}^-)\ \blacktriangleright\!\blacktriangleleft\ (s\bullet^-(\mathsf{IC}_x^-.\,(t\bowtie\pitchfork_{\alpha_0}^-)))
\end{array}
$$

Simulation of implication:

First let $u = \mathsf{IC}^-_\_ .\, \langle(\mathsf{IC}^-_\_ .\, \nu^- s), x'\rangle^-$ and note that:

$$\begin{array}{ll} & \mathbf{X'}_{u,z} \\ = & \pi_1^+(u \bullet^- \mathsf{IC}^+_{(\_:(A\Rightarrow B)\ominus)} .\, \mathsf{in}_1^+(\mathsf{IC}^+_{(\_:(\neg A)\ominus)} .\, \nu^+ z)) \\ \xrightarrow{\mathsf{beta}} & \pi_1^+(\langle(\mathsf{IC}^-_\_ .\, \nu^- s), x'\rangle^-) \\ \xrightarrow{\mathsf{proj}} & \mathsf{IC}^-_\_ .\, \nu^- s \end{array}$$

Hence:

$$\begin{array}{ll} & \mathbf{X}_u \\ = & \mathsf{IC}^+_{(z:A\ominus)} .\, (\mu^-(\mathbf{X'}_{u,z} \bullet^- \mathsf{IC}^+_{(\_:(\neg A)\ominus)} .\, \nu^+ z)) \bullet^+ z \\ \rightarrow^* & \mathsf{IC}^+_{(z:A\ominus)} .\, (\mu^-((\mathsf{IC}^-_\_ .\, \nu^- s) \bullet^- \mathsf{IC}^+_{(\_:(\neg A)\ominus)} .\, \nu^+ z)) \bullet^+ z \\ \xrightarrow{\mathsf{beta}} & \mathsf{IC}^+_{(z:A\ominus)} .\, (\mu^-(\nu^- s)) \bullet^+ z \\ \xrightarrow{\mathsf{neg}} & \mathsf{IC}^+_{(z:A\ominus)} .\, s \bullet^+ z \\ \xrightarrow{\mathsf{eta}} & s \end{array}$$

Hence:

$$\begin{array}{ll} & (\lambda^{\mathcal{C}}_x .\, t) \,@^{\mathcal{C}}\, s \\ = & \mathsf{IC}^+_{x'} . \\ & \quad \delta^+ \left(\begin{array}{l} (\mathsf{IC}^+_y .\, \mathsf{in}_2^+(t[x:=\mathbf{X}_y])) \\ \bullet^+ \mathsf{IC}^-_\_ .\, \langle(\mathsf{IC}^-_\_ .\, \nu^- s), x'\rangle^- \end{array} \right) \\ & \qquad [_{y'}.s \bowtie_{B^+} \mu^-(y' \bullet^+ \mathsf{IC}^-_\_ .\, \nu^- x')] \\ & \qquad [_{z'}.z' \bullet^+ x'] \\ \xrightarrow{\mathsf{beta}} & \mathsf{IC}^+_{x'} . \\ & \quad \delta^+ \ \mathsf{in}_2^+(t[x:=\mathbf{X}_u]) \\ & \qquad [_{y'}.s \bowtie_{B^+} \mu^-(y' \bullet^+ \mathsf{IC}^-_\_ .\, \nu^- x')] \\ & \qquad [_{z'}.z' \bullet^+ x'] \\ \xrightarrow{\mathsf{case}} & \mathsf{IC}^+_{x'} .\, t[x:=\mathbf{X}_u] \bullet^+ x' \\ \rightarrow^* & \mathsf{IC}^+_{x'} .\, t[x:=s] \bullet^+ x' \\ \xrightarrow{\mathsf{eta}} & t[x:=s] \end{array}$$

Computational content of the law of excluded middle:

Recall that $⫛^{\mathcal{C}}_A = ⫛^+_A$, where:

$$\begin{array}{rcl} ⫛^+_A & \stackrel{\mathrm{def}}{=} & \mathsf{IC}^+_{(x:(A\vee\neg A)\ominus)} .\, \mathsf{in}_2^+(\mathsf{IC}^+_{(y:\neg A\ominus)} .\, \nu^+ \pi_1^-(x \bullet^- \Delta^+_{y,A})) \\ \Delta^+_{y,A} & \stackrel{\mathrm{def}}{=} & \mathsf{IC}^+_{(\_:(A\vee\neg A)\ominus)} .\, \mathsf{in}_1^+(\mathsf{IC}^+_{(z:A\ominus)} .\, (y \bowtie_{A^+} \mathsf{IC}^+_{(\_:\neg A\ominus)} .\, \nu^+ z)) \end{array}$$

Let $u = \mathsf{IC}^{-}_{-}.\ \langle \updownarrow^{y'}_{x'} (s_1), \updownarrow^{y'}_{x'} (s_2)\rangle^{-}$. Then:

$$
\begin{array}{rl}
& \delta^{\mathcal{C}} \pitchfork^{\mathcal{C}}_{A}\ [_{x'}.s_1][_{x'}.s_2] \\
= & \mathsf{IC}^{+}_{y'}. \\
& \quad \delta^{+} \left(\begin{array}{l} \mathsf{IC}^{+}_{x}.\ \mathsf{in}^{+}_{2}(\mathsf{IC}^{+}_{y}.\ \nu^{+}\pi^{-}_{1}(x \bullet^{-} \Delta^{+}_{y,A})) \\ \bullet^{+} u \end{array} \right) \\
& \qquad [_{x'}.s_1 \bullet^{+} y'] \\
& \qquad [_{x'}.s_2 \bullet^{+} y'] \\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}^{+}_{y'}. \\
& \quad \delta^{+}\ \mathsf{in}^{+}_{2}(\mathsf{IC}^{+}_{y}.\ \nu^{+}\pi^{-}_{1}(u \bullet^{-} \Delta^{+}_{y,A})) \\
& \qquad [_{x'}.s_1 \bullet^{+} y'] \\
& \qquad [_{x'}.s_2 \bullet^{+} y'] \\
\xrightarrow{\mathsf{case}} & \mathsf{IC}^{+}_{y'}.\ s_2[x' := \mathsf{IC}^{+}_{y}.\ \nu^{+}\pi^{-}_{1}(u \bullet^{-} \Delta^{+}_{y,A})] \bullet^{+} y' \\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}^{+}_{y'}.\ s_2[x' := \mathsf{IC}^{+}_{-}.\ \nu^{+}\pi^{-}_{1}(\langle \updownarrow^{y'}_{x'} (s_1), \updownarrow^{y'}_{x'} (s_2)\rangle^{-})] \bullet^{+} y' \\
\xrightarrow{\mathsf{proj}} & \mathsf{IC}^{+}_{y'}.\ s_2[x' := \mathsf{IC}^{+}_{-}.\ \nu^{+}\updownarrow^{y'}_{x'} (s_1)] \bullet^{+} y' \\
= & \mathsf{IC}^{+}_{y'}.\ s_2[x' := \mathsf{IC}^{+}_{-}.\ \nu^{+}(\mathsf{IC}^{-}_{x'}.\ s_1 \bowtie y')] \bullet^{+} y'
\end{array}
$$

□

6. SECOND ORDER PRK

In this chapter we extend the results from previous chapters to second order logic. In particular, we extend λ^{PRK} to λ_2^{PRK}, a system including universal and existential quantifiers. Then we prove subject reduction (Section 1), we prove confluence (Section 2), and we characterize normal forms (Section 3). Finally, we show that λ_2^{PRK} is conservative with respect to classical second order logic (Section 4), and that second order logic may be embedded in λ_2^{PRK} (Section 5).

We have attempted, unsuccessfully, to prove termination of this calculus using a simulation strategy, as done on Section 2.3. The problem is that System F extended with recursive type constraints does not seem to be expressive enough to work as the target of the translation. The question of whether λ_2^{PRK} is strongly normalizing is left as an open problem.

Definition 52 (The second order system λ_2^{PRK})**.** The λ_2^{PRK}-calculus extends the λ^{PRK}-calculus with the following types, terms, typing rules, and rewriting rules.

The set of *types* is extended as follows:

$$\begin{array}{rcll} A & ::= & \dots & \\ & | & \forall\alpha.\, A & \text{universal quantification} \\ & | & \exists\alpha.\, A & \text{existential quantification} \end{array}$$

The syntax of *terms* is extended as follows:

$$\begin{array}{rcll} t & ::= & \dots & \\ & | & \lambda^{\pm}\alpha.\, t & \forall^+/\exists^- \text{ introduction} \\ & | & t \bullet^{\pm} A & \forall^+/\exists^- \text{ elimination} \\ & | & \langle A, t\rangle^{\pm} & \exists^+/\forall^- \text{ introduction} \\ & | & \nabla^{\pm}(\alpha, x).t.s & \exists^+/\forall^- \text{ elimination} \end{array}$$

The following eight *typing rules* are added:

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \alpha \notin \mathsf{fv}(\Gamma)}{\Gamma \vdash \lambda^{+}\alpha.\, t : (\forall\alpha.\, A)^{+}}\,\mathrm{I}\forall^{+} \qquad \frac{\Gamma \vdash t : A^{\ominus} \qquad \alpha \notin \mathsf{fv}(\Gamma)}{\Gamma \vdash \lambda^{-}\alpha.\, t : (\exists\alpha.\, A)^{-}}\,\mathrm{I}\exists^{-}$$

$$\frac{\Gamma \vdash t : (\forall\alpha.\, B)^{+}}{\Gamma \vdash t \bullet^{+} A : B^{\oplus}[\alpha := A]}\,\mathrm{E}\forall^{+} \qquad \frac{\Gamma \vdash t : (\exists\alpha.\, B)^{-}}{\Gamma \vdash t \bullet^{-} A : B^{\ominus}[\alpha := A]}\,\mathrm{E}\exists^{-}$$

$$\frac{\Gamma \vdash t : B^{\oplus}[\alpha := A]}{\Gamma \vdash \langle A, t\rangle^{+} : (\exists\alpha.\, B)^{+}}\,\mathrm{I}\exists^{+} \qquad \frac{\Gamma \vdash t : B^{\ominus}[\alpha := A]}{\Gamma \vdash \langle A, t\rangle^{-} : (\forall\alpha.\, B)^{-}}\,\mathrm{I}\forall^{-}$$

$$\frac{\Gamma \vdash t : (\exists\alpha.\, A)^{+} \qquad \Gamma, x : A^{\oplus} \vdash s : P \qquad \alpha \notin \mathsf{fv}(\Gamma, P)}{\Gamma \vdash \nabla^{+}(\alpha, x).t.s : P}\,\mathrm{E}\exists^{+}$$

$$\frac{\Gamma \vdash t : (\forall\alpha.\, A)^{-} \qquad \Gamma, x : A^{\ominus} \vdash s : P \qquad \alpha \notin \mathsf{fv}(\Gamma, P)}{\Gamma \vdash \nabla^{-}(\alpha, x).t.s : P}\,\mathrm{E}\forall^{-}$$

The following *rewriting rules* are added:

$$\begin{array}{rcl}
(\lambda^{\pm}\alpha.\, t) \bullet^{\pm} A & \xrightarrow{\mathsf{appT}} & t[\alpha := A] \\
\nabla^{\pm}(\alpha, x).\langle A, t\rangle^{\pm}.s & \xrightarrow{\mathsf{open}} & s[\alpha := A][x := t] \\
(\lambda^{+}\alpha.\, t) \Join \langle A, s\rangle^{-} & \xrightarrow{\mathsf{absLamPairT}} & t[\alpha := A] \bowtie s \\
\langle A, t\rangle^{+} \Join (\lambda^{-}\alpha.\, s) & \xrightarrow{\mathsf{absPairLamT}} & t \bowtie s[\alpha := A]
\end{array}$$

In the rest of this chapter, unless explicited otherwise, the types, terms, typing rules, and rewriting rules refer to λ_2^{PRK}.

1 Subject Reduction

Again, we follow the development done in Prop. 31, and we only show that the new reduction rules hold at the root of the evaluation context (since by IH any reduction inside of a term would keep this property valid).

However, we first need to show the admissibility of a type substitution rule, *SubT*, where we define the substitution on propositions and on typing contexts as expected.

Lemma 53. *The following rule is admissible on our system.*

$$\frac{\Gamma \vdash t : P}{\Gamma[\alpha := A] \vdash t[\alpha := A] : P[\alpha := A]}\ \textsc{SubT}$$

Proof. Routinary by induction on the derivation. □

Theorem 54 (Second Order Subject Reduction)**.** *If $\Gamma \vdash t : P$ and $t \to s$, then $\Gamma \vdash s : P$.*

Proof. Extends the proof of Prop. 31

- **Application** ($\xrightarrow{\mathsf{appT}}$):

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : B^{\oplus}} \quad \alpha \notin \mathsf{fv}(\Gamma)}{\Gamma \vdash \lambda^{+}\alpha.\, t : \forall\alpha.\, B^{+}}\ \mathrm{I}\forall^{+}}{\Gamma \vdash (\lambda^{+}\alpha.\, t) \bullet^{+} A : B^{\oplus}[\alpha := A]}\ \mathrm{E}\forall^{+} \quad \xrightarrow{\mathsf{appT}} \quad \dfrac{\dfrac{\pi}{\Gamma \vdash t : B^{\oplus}}}{\Gamma \vdash t[\alpha := A] : B^{\oplus}[\alpha := A]}\ \textsc{SubT}\ (\text{Lem. 53})$$

Note that $\Gamma = \Gamma[\alpha := A]$ since $\alpha \notin \mathsf{fv}(\Gamma)$.

- **Unpacking** ($\xrightarrow{\mathsf{open}}$):

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : B^{\oplus}[\alpha := A]}}{\Gamma \vdash \langle A, t\rangle^{\pm} : (\exists \alpha. B)^{+}}\,\mathrm{I}\exists^{+} \qquad \dfrac{\pi'}{\Gamma, x : B^{\oplus} \vdash s : P} \qquad \alpha \notin \mathsf{fv}(\Gamma, P)}{\Gamma \vdash \nabla^{+}(\alpha, x).\langle A, t\rangle^{+}.s : P}\,\mathrm{E}\exists^{+}$$

$$\xrightarrow{\mathsf{open}}$$

$$\dfrac{\dfrac{\pi}{\Gamma \vdash t : B^{\oplus}[\alpha := A]} \qquad \dfrac{\dfrac{\pi'}{\Gamma, x : B^{\oplus} \vdash s : P}}{\Gamma, x : B^{\oplus}[\alpha := A] \vdash s[\alpha := A] : P}\,\textsc{SubT}\text{ (Lem. 53)}}{\Gamma \vdash s[\alpha := A][x := t] : P}\,\textsc{Sub}$$

- **Absurdity Abstraction-Pair** ($\xrightarrow{\mathsf{absLamPairT}}$):

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : B^{\oplus}} \quad \alpha \notin \mathsf{fv}(\Gamma)}{\Gamma \vdash \lambda^{+}\alpha.\, t : \forall \alpha.\, B^{+}}\,\mathrm{I}\forall^{+} \qquad \dfrac{\dfrac{\pi'}{\Gamma \vdash s : B^{\ominus}[\alpha := A]}}{\Gamma \vdash \langle A, s\rangle^{-} : \forall \alpha.\, B^{-}}\,\mathrm{I}\forall^{-}}{\Gamma \vdash (\lambda^{+}\alpha.\, t) \blacktriangleright\!\!\blacktriangleleft_{P} \langle A, s\rangle^{-} : P}\,\textsc{Abs}$$

$$\xrightarrow{\mathsf{absLamPairT}}$$

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : B^{\oplus}}}{\Gamma \vdash t[\alpha := A] : B^{\oplus}[\alpha := A]}\,\textsc{SubT}\text{ (Lem. 53)} \qquad \dfrac{\pi'}{\Gamma \vdash s : B^{\ominus}[\alpha := A]}}{\Gamma \vdash t[\alpha := A] \bowtie_{P} s : P}\,\textsc{Abs}$$

- **Absurdity Pair-Abstraction** ($\xrightarrow{\mathsf{absPairLamT}}$):
  Symmetric to $\xrightarrow{\mathsf{absLamPairT}}$.

□

2 Confluence

The λ_2^{PRK}-calculus enjoys confluence, and it can be proved by extending the higher-order rewriting system presented on Appendix 3.

Proposition 55. *The λ_2^{PRK}-calculus is confluent.*

Proof. The rewriting system λ^{PRK} can be modeled as a higher-order rewriting system (HRS) in the sense of Nipkow. This HRS is *orthogonal*, *i.e.* left-linear without critical pairs, which entails that it is confluent [10]. □

3 Characterization of Normal Forms

In this section we will provide a characterization of Normal Forms for λ_2^{PRK}, in a similar way as we did in Section 2.4. The proofs are similar to those in Section 2.4, so we assume most of the heavy work already done, and we limit ourselves to completing the missing pieces.

First we provide an inductive characterization of the set of **normal forms** of λ_2^{PRK}.

Definition 56 (Second order normal terms)**.** The sets of *normal terms* $(N, \ldots)$ and *neutral terms* $(S, \ldots)$ are defined mutually inductively by extending Def. 41 in the following manner:

$$\begin{array}{rcl} N & ::= & \ldots \\ & | & \lambda^{\pm}\alpha.\, N \\ & | & \langle A, N\rangle^{\pm} \\ \\ S & ::= & \ldots \\ & | & S \bullet^{\pm} A \\ & | & \nabla^{\pm}(\alpha, x).S.N \end{array}$$

Proposition 57. *A term is normal if and only if it does not reduce in* λ_2^{PRK}.

Proof. ($\Rightarrow$) Let t be a normal term, and let us check that it is a $\to$-normal form. We proceed by structural induction on t.

The cases corresponding to introduction rules are straightforward by IH. For example, if $t = \lambda^{\pm}\alpha.\, N$, then by IH N has no $\to$-redexes. Moreover, there are no rules involving a type abstraction $\lambda^{\pm}\alpha.\, N$ at the root, so $\lambda^{\pm}\alpha.\, N$ is in $\to$-normal form.

The cases corresponding to elimination rules and the absurdity rule are also straightforward by IH, observing that there cannot be a redex at the root. For example, if $t = \nabla^{\pm}(\alpha, x).S.N$, then by IH S has no $\to$-redexes. Moreover, the only rule involving a type opening $\nabla^{\pm}(\alpha, x).S.N$ at the root is open, which would require that $S = \langle A, t\rangle^{\pm}$. But this is impossible —as can be checked by exhaustive case analysis on S—, so t is in $\to$-normal form.

($\Leftarrow$) Let t be a $\to$-normal form, let us check that it is a normal term. We proceed by induction on the structure of the term t, extending the results from Prop. 42:

- **Type abstraction** ($\lambda^{\pm}\alpha.\, t$): by IH, t is a normal term, and that is enough.
- **Type application** ($t \bullet^{\pm} A$): by IH, t is a normal term, if it is also neutral, then we are done. It suffices to show that it cannot be a normal but not neutral term: if it were then t would need to be of the form $\lambda^{\pm}\alpha.\, N$, since it would be the only possible term with the right type $\forall\alpha.\, A^{\pm}$. Then the rule appT may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
- **Type packing** ($\langle A, t\rangle^{\pm}$): by IH, t is a normal term, then we are done.
- **Type unpacking** ($\nabla^{\pm}(\alpha, x).t.s$): by IH t and s are normal terms. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t must be of the form $\exists\alpha.\, A^{\pm}$, we have that t is of the form $\langle B, t'\rangle^{\pm}$. Then the rule open may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.

□

Next, we adapt the canonicity results, briefly recalling the definitions and expanding them to λ_2^{PRK}. A term is *canonical* if it has any of the following shapes:

$$\langle t_1, t_2\rangle^{\pm} \quad \mathsf{in}_i^{\pm}(t) \quad \nu^{\pm}t \quad \mathsf{IC}_x^{\pm}.\,t \quad \lambda^{\pm}\alpha.\,t \quad \langle A, t\rangle^{\pm}$$

A typing context is *classical* if all the assumptions are classical. A *case-context* is a context of the form $\mathtt{K} ::= \Box \mid \delta^{\pm}\mathtt{K}\,[_x.t][_y.s] \mid \nabla^{\pm}(\alpha, x).\mathtt{K}.s$. An *eliminative context* is a context of the form $\mathtt{E} ::= \Box \mid \pi_i^{\pm}(\mathtt{E}) \mid \mu^{\pm}\mathtt{E} \mid \mathtt{E}\bullet^{\pm} A \mid \mathtt{K}\langle\mathtt{E}\rangle$. Note that $\Box \bullet^{\pm} t$ is not eliminative, but $\Box \bullet^{\pm} A$ is, and that all case-contexts are eliminative. An *explosion* is a term of the form $t \bowtie_P s$ or of the form $t \bullet^{\pm} s$. A term is *closed* if it has no free variables. A term is *open* if it not closed, *i.e.* it has at least one free variable.

Theorem 58 (Canonicity)**.** *Extending Thm. 43*

1. *Let $\vdash t : P$ where t is a normal form. Then t is canonical.*
2. *Let $\Gamma \vdash t : A^{\pm}$ where Γ is classical and t is a normal form. Then either t is canonical or t is of the form $\mathtt{K}\langle t'\rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion.*
3. *Let $\Gamma \vdash t : A^{\oplus}$ or $\Gamma \vdash t : A^{\ominus}$, where Γ is classical and t is a normal form. Then either $t = \mathsf{IC}_x^{\pm}.\,t'$ or $t = \mathtt{E}\langle t''\rangle$, where $\mathtt{E}$ is an eliminative context and t'' is a variable or an open explosion.*

Proof.

1. Same as Thm. 43.
2. Let $\Gamma \vdash t : P$ where Γ is classical and t is a normal form. By Prop. 57 either t is canonical or it is a neutral term. If t is canonical we are done. If t is a neutral term it suffices to show the following claim, namely that if $\Gamma \vdash t : B^{\pm}$ is a derivable judgment such that Γ is classical and t is a neutral term, then t is of the form $t = \mathtt{K}\langle t'\rangle$, where $\mathtt{K}$ is a case-context and t' is an open explosion. Most of the cases are the same as in Thm. 43, we show the two new ones:
   - **Type application** ($S \bullet^{\pm} A$): this case is impossible, as $\Gamma \vdash S \bullet^{\pm} A : P$ where P must be of the form $C^{\oplus}$ or $C^{\ominus}$, hence P cannot be of the form $B^{\pm}$.
   - **Type unpacking** ($\nabla^{\pm}(\alpha, x).S.N$): by inversion of the typing rules we have that either $\Gamma \vdash S : (\exists\beta.\,A)^{+}$ or $\Gamma \vdash S : (\forall\beta.\,A)^{-}$. In both cases we may apply the IH to conclude that S is of the form $S = \mathtt{K}\langle t'\rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \nabla^{\pm}(\alpha, x).(\mathtt{K}\langle t'\rangle).N$ where now $\nabla^{\pm}(\alpha, x).(\mathtt{K}).N$ is a case-context.
3. Let $\Gamma \vdash t : A^{\oplus}$ or $\Gamma \vdash t : A^{\ominus}$, where Γ is classical and t is a normal form. By Prop. 57 either t is canonical or it is a neutral term. If t is canonical, then by the constraints on its type it must be of the form $t = \mathsf{IC}_x^{\pm}.\,t'$, so we are done. If t is neutral, it suffices to show the following claim namely that if $\Gamma \vdash t : P$ is a derivable judgment, with $P \in \{B^{\oplus}, B^{\ominus}\}$, such that Γ is classical and t is a neutral term, then t is of the form $t = \mathtt{E}\langle t'\rangle$, where $\mathtt{E}$ is an eliminative context and t' is a variable or an open explosion. Most cases are the same as in Thm. 43, we show the two new ones:

- **Type application** ($S \bullet^{\pm} A$): by inversion of the typing rules, we have that either $\Gamma \vdash S : (\forall\alpha.\, B)^{+}$ or $\Gamma \vdash S : (\exists\alpha.\, B)^{-}$. In both cases we may apply the second item of this lemma to conclude that S is of the form $S = \mathtt{K}\langle t'\rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \mathtt{K}\langle t'\rangle \bullet^{\pm} A$, where now $\mathtt{K} \bullet^{\pm} A$ is an eliminative context.
- **Type unpacking** ($\nabla^{\pm}(\alpha, x).S.N$): by inversion of the typing rules we have that either $\Gamma \vdash S : (\exists\beta.\, A)^{+}$ or $\Gamma \vdash S : (\forall\beta.\, A)^{-}$. In both cases we may apply the second item of this lemma to conclude that S is of the form $S = \mathtt{K}\langle t'\rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \nabla^{\pm}(\alpha, x).(\mathtt{K}\langle t'\rangle).N$ where now $\nabla^{\pm}(\alpha, x).(\mathtt{K}).N$ is an eliminative context.

□

4 Conservativity

As in Prop. 49 we show that λ_2^{PRK} is conservative with respect to classical second order logic, via a translation of judgments from λ_2^{PRK} into classical second order logic, and we show that every derivable judgment in λ_2^{PRK} is mapped to a valid classical proposition.

Theorem 59 (Conservativity of λ_2^{PRK})**.** *Define $c(-)$ as in Prop. 49:*

$$\begin{array}{llll} c(A^{\oplus}) \stackrel{\mathrm{def}}{=} A & c(A^{\ominus}) \stackrel{\mathrm{def}}{=} \neg A \\ c(A^{+}) \stackrel{\mathrm{def}}{=} A & c(A^{-}) \stackrel{\mathrm{def}}{=} \neg A \end{array}$$

If $P_1, \ldots, P_n \vdash Q$ is derivable, then $c(P_1), \ldots, c(P_n) \vdash c(Q)$ is a classically valid proposition.

Proof. The proof is straightforward by showing the admissibility of rules by induction on the derivation of $P_1, \ldots, P_n \vdash Q$.

For instance, if the last typing step is $\mathrm{I}\exists^{-}$, then $Q = (\exists\alpha.\, A)^{-}$ and:

$$\frac{P_1, \ldots, P_n \vdash A^{\ominus} \qquad \alpha \notin \mathsf{fv}(P_1, \ldots, P_n)}{P_1, \ldots, P_n \vdash (\exists\alpha.\, A)^{-}}\ \mathrm{I}\exists^{-}$$

By IH we know that $c(P_1), \ldots, c(P_n) \vdash \neg A$, and that $\alpha \notin \mathsf{fv}(c(P_1), \ldots, c(P_n))$, since $c(P)$ doesn't change the free variables of P.

Therefore, we can show, in second order logic, that $c(P_1), \ldots, c(P_n) \vdash \forall\alpha.\, \neg A$, which is classically equivalent to $c(P_1), \ldots, c(P_n) \vdash \neg\exists\alpha.\, A$, which is exactly what we were after: $c(P_1), \ldots, c(P_n) \vdash c((\exists\alpha.\, A)^{-})$.

□

5 Embedding

To show that classical second order logic may be embedded in λ_2^{PRK}, we extend the results of Thm. 50 by showing the admissibility of all the inference schemes, laid out on Appendix 1.1.

Theorem 60 (Second Order Completeness)**.** *If A is a classically valid proposition, then $\vdash A^{\oplus}$ is derivable.*

Proof. In general, if $A_1, \ldots, A_n \vdash B$ is derivable in natural deduction with the law of excluded middle and universal and existential quantification, then $x_1 : {A_1}^{\oplus}, \ldots, x_n : {A_n}^{\oplus} \vdash B^{\oplus}$ is derivable in our second order system. It suffices to show that the extra inference schemes are admissible.

- **Universal Quantification Introduction** ($\mathrm{I}\forall_{ND}$):
  Let:
  $$\Gamma \vdash t : A^{\oplus} \qquad \alpha \notin \mathsf{fv}(\Gamma)$$
  We define:
  $$\lambda^{\mathcal{C}}_{\alpha}.\, t \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{\_:(\forall\alpha.\, A)^{\ominus}}.\, \lambda^{+}\alpha.\, t$$
  Then $\Gamma \vdash \lambda^{\mathcal{C}}_{\alpha}.\, t : \forall\alpha.\, A^{\oplus}$.

- **Universal Quantification Elimination** ($\mathrm{E}\forall_{ND}$):
  Let
  $$\Gamma \vdash t : \forall\alpha.\, B^{\oplus}$$
  We define:
  $$t \bullet^{\mathcal{C}} A \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{x:B[\alpha=A]^{\ominus}}.\, ((t \bullet^{+} \mathsf{IC}^{-}_{\_:\forall\alpha.\, B^{\oplus}}.\, \langle A, x\rangle^{-}) \bullet^{+} A) \bullet^{+} x$$
  Then $\Gamma \vdash t \bullet^{\mathcal{C}} A : B[\alpha := A]^{\oplus}$.

- **Existential Quantification Introduction** ($\mathrm{I}\exists_{ND}$):
  Let
  $$\Gamma \vdash t : B[\alpha := A]^{\oplus}$$
  And define:
  $$\langle A, t\rangle^{\mathcal{C}} \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{\_:\exists\alpha.\, B^{\ominus}}.\, \langle A, t\rangle^{+}$$
  Then $\Gamma \vdash \langle A, t\rangle^{\mathcal{C}} : \exists\alpha.\, B^{\oplus}$.

- **Existential Quantification Elimination** ($\mathrm{I}\exists_{ND}$):
  Let
  $$\Gamma \vdash t : \exists\alpha.\, A^{\oplus} \qquad \Gamma, x : A^{\oplus} \vdash s : B^{\oplus} \qquad \alpha \notin \mathsf{fv}(\Gamma, B)$$
  Then, we define:
  $$\nabla^{\mathcal{C}}(\alpha, x).t.s \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{y:B^{\ominus}}.\, (\nabla^{+}(\alpha, x).t \bullet^{+} \mathsf{IC}^{-}_{\_:\exists\alpha.\, A^{\oplus}}.\, \lambda^{-}\alpha.\, \updownarrow^{y}_{x} (s).s) \bullet^{+} y$$
  Finally, $\Gamma \vdash \nabla^{\mathcal{C}}(\alpha, x).t.s : B^{\oplus}$.

$\square$

7. CONCLUSION

In this thesis we have introduced the logical system PRK, a proof system formulated in natural deduction style, distinguishing between strong and classical propositions, and between proofs and refutations of propositions. The system has been studied in depth.

First, we have formulated a notion of Kripke semantics for system PRK. This provides a deeper understanding of how PRK treats classicality: a classical formula can be considered valid if its refutation makes the world collapse (*i.e.* makes the context inconsistent).

The main interest point of PRK is that it can be given a well behaved computational interpretation. To this aim, we have defined λ^{PRK}, a calculus whose type system corresponds to PRK, and we have demonstrated that it enjoys subject reduction, confluence, strong normalization, and a notion of canonical proof. The proof of strong normalization, even if not particularly innovative, is nontrivial, and it is based on a translation of λ^{PRK} into System F extended with recursive equations between types. System F extended with these recursive equations is strongly normalizing as a consequence of the work by Mendler [11], because we only require *non-strictly positive recursion.* The translation does not erase reduction steps, which means that λ^{PRK} is also strongly normalizing.

We have studied the relationship between PRK and classical logic. In particular, we have shown that every valid formula in PRK can be projected to a valid formula in classical logic (via a forgetful mapping) and, conversely, every classically valid formula can be embedded into PRK. What is more, this embedding has an interesting computational interpretation when considered as terms of $\lambda_{\eta}^{\text{PRK}}$, an extension of λ^{PRK} with η-like reduction.

Finally, we have opened up the field of study for PRK by showing an extension to second order logic, introducing second-order universal and existential quantifiers, and we have presented a calculus for it, dubbed λ_2^{PRK}. This extension has been shown to enjoy most of the properties that were proved for λ^{PRK} other than strong normalization, which remains as an open conjecture.

We hope that this foundational work on PRK may stimulate other people to keep investigating these ideas.

1 Related Work

Constructible falsity The idea of defining negation by means of constructive refutations, as opposed to negation defined by reduction to absurdity, was already studied over 70 years ago by Nelson [6]. As stated on the introduction, our work can be understood as the result of adding an extra axis across types (formulas) besides the positive vs. negative distinction, the distinction between strong vs. classical types. In spite of the similarities between Nelson's work and our own, it is worthy to mention that Nelson did not have the goal in mind to formulate a logic in which classical logic could be embedded.

Other computational interpretations of classical logic The quest for the computational meaning behind classical logic is not a novel point of this work. A lot of research has been previously done towards this goal. In particular we can mention Parigot's $\lambda\mu$-calculus [2], Barbanera and Berardi's symmetric lambda calculus (λ_{Prop}^{Sym}) [3], and Curien and Herbelin's $\bar{\lambda}\mu\tilde{\mu}$-calculus [4]. These calculi have been heavily studied, and many of the desired properties (confluence, strong normalization, etc.) have been proved to hold.

A more in-depth study of the relationship between these calculi and λ^{PRK} could be of interest and we leave it as future work. We believe that our work offers a different perspective. In particular, by using a form of constructible falsity, as presented by Nelson[6], we provide a calculus with an explicit, and clear, duality between proofs and refutations, that moves away from the duality between values and continuations presented by Parigot, and by Curien and Herbelin; or even from the syntactic approach to handle negation used by Barbanera and Berardi.

Kripke semantics for classical logic Our definition of Kripke model is slightly different from that found on other works, but it shares many traits with the one presented by Ilik, Lee, and Herbelin [9] to provide a Kripke semantics for classical logic. Both notions of semantics consider differentiated sets of positive and negative base variables, and build up from these.

2 Future work

We believe PRK has the potential to become a fruitful source for future work, we outline some potential ideas below.

Second order logic The first, obvious, line of work is to complete the study of strong normalization for the second order version of the calculus, λ_2^{PRK}, presented in Section 6. In particular, we have no reason to believe this calculus is not strongly normalizing, but the proof techniques that we used for λ^{PRK} do not appear to be directly applicable.

Kripke Models Our notion of Kripke semantics, as presented on Section 3, could be studied by itself, as a notion of Kripke semantics for classical logic. In particular, a relationship between our notion of model and the one of Ilik et al. [9] may be established.

Sequent calculus Although natural deduction, as a framework to define a logical system, is better suited to our goal of studying PRK from the point of view of the propositions-as-types correspondence, a sequent calculus system for PRK could offer an interesting alternative point of view of the system, specially given that PRK has some pleasant symmetries proper of sequent calculi.

8. APPENDIX

1 Natural Deduction for classical logic

Definition 61 (Natural deduction for classical logic)**.** Propositions are given by:

$$A ::= \alpha \mid A \wedge A \mid A \vee A \mid \neg A$$

We write $\bot$ for $(\alpha_0 \wedge \neg\alpha_0)$, where α_0 is some fixed propositional variable. Inference rules are given by:

$$\frac{}{\Gamma, A \vdash A}\ \mathrm{Ax}_{ND}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \mathrm{I}\wedge_{ND} \qquad \frac{\Gamma \vdash A_1 \wedge A_2 \quad i \in \{1,2\}}{\Gamma \vdash A_i}\ \mathrm{E}\wedge_{iND}$$

$$\frac{\Gamma \vdash A_i \quad i \in \{1,2\}}{\Gamma \vdash A_1 \vee A_2}\ \mathrm{I}\vee_{iND} \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ \mathrm{E}\vee_{ND}$$

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ \mathrm{I}\neg_{ND} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash \bot}\ \mathrm{E}\neg_{ND}$$

$$\frac{}{\Gamma \vdash A \vee \neg A}\ \mathrm{LEM}_{ND}$$

1.1 ... and its extension to second order

Def. 61 is extended to second order logic as follows:

Definition 62 (Extension of Natural Deduction for Second Order Classical Logic)**.** The set of propositions is extended as follows:

$$A \quad ::= \quad \ldots \quad \mid \quad \forall\alpha.\, A \quad \mid \quad \exists\alpha.\, A$$

The set of inference rules is extended as follows:

$$\frac{\Gamma \vdash A \quad \alpha \notin \mathsf{fv}(\Gamma)}{\Gamma \vdash \forall\alpha.\, A}\ \mathrm{I}\forall_{ND} \qquad \frac{\Gamma \vdash \forall\alpha.\, B}{\Gamma \vdash B[\alpha := A]}\ \mathrm{E}\forall_{ND}$$

$$\frac{\Gamma \vdash B[\alpha := A]}{\Gamma \vdash \exists\alpha.\, B}\ \mathrm{I}\exists_{ND} \qquad \frac{\Gamma \vdash \exists\alpha.\, A \quad \Gamma, A \vdash P \quad \alpha \notin \mathsf{fv}(\Gamma, P)}{\Gamma \vdash P}\ \mathrm{E}\exists_{ND}$$

2 System F with Mendler recursion

2.1 System F Extended with Recursive Type Constraints

Definition 63 (Extended System F)**.** The set of *types* is given by:

$$A, B, \ldots ::= \alpha \mid A \to B \mid \forall\alpha.A$$

The set of *terms* is given by:

$$t, s, \ldots ::= x \mid \lambda x^A.\, t \mid t\, s \mid \lambda\alpha.\, t \mid t\, A$$

we omit type annotations over variables when clear from the context. A *type constraint* is an equation of the form $\alpha \equiv A$. Each set $\mathcal{C}$ of type constraints induces a notion of equivalence between types, written $A \equiv B$ and defined as the congruence generated by $\mathcal{C}$. More precisely:

$$\frac{(A \equiv B) \in \mathcal{C}}{A \equiv B}\ \textsc{constr} \qquad \frac{}{A \equiv A}\ \textsc{refl} \qquad \frac{A \equiv B}{B \equiv A}\ \textsc{sym}$$

$$\frac{A \equiv B \quad B \equiv C}{A \equiv C}\ \textsc{trans} \qquad \frac{A \equiv B}{C[\alpha := A] \equiv C[\alpha := B]}\ \textsc{cong}$$

We suppose that $\mathcal{C}$ is fixed. Typing judgments are of the form $\Gamma \vdash t : A$.

$$\frac{}{\Gamma, x : A \vdash x : A}\ \textsc{Ax} \qquad \frac{\Gamma \vdash t : A \quad A \equiv B}{\Gamma \vdash t : B}\ \textsc{Conv}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.\, t : A \to B}\ \mathrm{I}{\to} \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash s : A}{\Gamma \vdash t\, s : B}\ \mathrm{E}{\to}$$

$$\frac{\Gamma \vdash t : A \quad \alpha \notin \mathsf{fv}(\Gamma)}{\Gamma \vdash \lambda\alpha.\, t : \forall\alpha.A}\ \mathrm{I}\forall \qquad \frac{\Gamma \vdash t : \forall\alpha.A}{\Gamma \vdash t\, B : A[\alpha := B]}\ \mathrm{E}\forall$$

Reduction is defined as the closure by arbitrary contexts of the following rewriting rules:

$$\begin{array}{rcl} (\lambda x.\, t)\, s & \to & t[x := s] \\ (\lambda\alpha.\, t)\, A & \to & t[\alpha := A] \end{array}$$

Definition 64 (Positive/negative occurrences)**.** The set of type variables occurring positively (resp. negatively) in a type A are written $\mathsf{p}(A)$ (resp. $\mathsf{n}(A)$) and defined by:

$$\begin{array}{rclrcl} \mathsf{p}(\alpha) & \stackrel{\text{def}}{=} & \{\alpha\} & \mathsf{n}(\alpha) & \stackrel{\text{def}}{=} & \varnothing \\ \mathsf{p}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{n}(A) \cup \mathsf{p}(B) & \mathsf{n}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{p}(A) \cup \mathsf{n}(B) \\ \mathsf{p}(\forall\alpha.A) & \stackrel{\text{def}}{=} & \mathsf{p}(A) \setminus \{\alpha\} & \mathsf{n}(\forall\alpha.A) & \stackrel{\text{def}}{=} & \mathsf{n}(A) \setminus \{\alpha\} \end{array}$$

Definition 65 (Positivity condition)**.** A set of type constraints $\mathcal{C}$ verifies the *positivity condition* if for every type constraint $(\alpha \equiv A) \in \mathcal{C}$ and every type B such that $\alpha \equiv B$ one has that $\alpha \notin \mathsf{n}(B)$.

Theorem 66 (Mendler)**.** *If $\mathcal{C}$ verifies the positivity condition, then System F extended with the recursive type constraints $\mathcal{C}$ is strongly normalizing.*

Proof. See [11, Theorem 13]. □

Abbreviations. We define the following standard abbreviations for types:

$$\begin{array}{rcl} \mathbf{1} & \stackrel{\text{def}}{=} & \forall\alpha.(\alpha \to \alpha) \\ \mathbf{0} & \stackrel{\text{def}}{=} & \forall\alpha.\alpha \\ \neg A & \stackrel{\text{def}}{=} & A \to \mathbf{0} \\ A \times B & \stackrel{\text{def}}{=} & \forall\alpha.((A \to B \to \alpha) \to \alpha) \\ A + B & \stackrel{\text{def}}{=} & \forall\alpha.((A \to \alpha) \to (B \to \alpha) \to \alpha) \end{array}$$

And the following terms. We omit the typing contexts for succintness:

$$\begin{array}{rcl}
\star & \stackrel{\text{def}}{=} & \lambda\alpha.\,\lambda x^{\alpha}.\,x \\
 & : & \mathbf{1} \\[1em]
\mathcal{E}_A(t) & \stackrel{\text{def}}{=} & t\,A \\
 & : & A \\
 & & \text{if } t : \mathbf{0} \\[1em]
\langle t, s\rangle & \stackrel{\text{def}}{=} & \lambda\alpha.\,\lambda f^{A\to B\to\alpha}.\,f\,t\,s \\
 & : & A \times B \\
 & & \text{if } t : A \text{ and } s : B \\[1em]
\pi_i(t) & \stackrel{\text{def}}{=} & t\,A_i\,(\lambda x_1^{A_1}.\,\lambda x_2^{A_2}.\,x_i) \\
 & : & A_i \\
 & & \text{if } t : A_1 \times A_2 \\[1em]
\mathsf{in}_i(t) & \stackrel{\text{def}}{=} & \lambda\alpha.\,\lambda f_1^{A_1\to\alpha}.\,\lambda f_2^{A_2\to\alpha}.\,f_i\,t \\
 & : & A_1 + A_2 \\
 & & \text{if } t : A_i \text{ and } i \in \{1,2\} \\[1em]
\delta t\,[_{x:A_1}.s_1][_{x:A_2}.s_2] & \stackrel{\text{def}}{=} & t\,B\,(\lambda x^{A_1}.\,s_1)\,(\lambda x^{A_2}.\,s_2) \\
 & : & B \\
 & & \text{if } t : A_1 + A_2 \text{ and } s_i : B \\
 & & \text{for each } i \in \{1,2\}
\end{array}$$

3 Higher-Order Rewriting System formalization

Encoding λ^{PRK} as a Higher-Order Rewriting System allows us to prove confluence easily, by showing that is *orthogonal*, *i.e.* left-linear without critical pairs, which entails that it is confluent [10].

On top of that, an increasing number of automated tools to prove different properties on these systems exist, by providing a model of our development on a standard language we make it easier to experiment with it.

The model is presented in the format used at the Confluence Competition [14], and the different tools can be tried online [15][1].

```
(FUN
  pi1p : a -> a
  pi2p : a -> a
  pairp : a -> a -> a
  in1p  : a -> a
  in2p  : a -> a
  deltap : a -> (a -> a) -> (a -> a) -> a
  nup : a -> a
  mup : a -> a
  icp : (a -> a) -> a
  ecp : a -> a -> a
```

[1] We recomend the 2020 HRS `SOL` tool.

```
  pi1n : a -> a
  pi2n : a -> a
  pairn : a -> a -> a
  in1n  : a -> a
  in2n  : a -> a
  deltan : a -> (a -> a) -> (a -> a) -> a
  nun : a -> a
  mun : a -> a
  icn : (a -> a) -> a
  ecn : a -> a -> a

  abs : a -> a -> a
)

(VAR
  f : a -> a
  g : a -> a
  t : a
  s : a
  u : a
  t1 : a
  t2 : a
)

(RULES
  pi1p (pairp t1 t2) -> t1,
  pi2p (pairp t1 t2) -> t2,
  deltap (in1p t) f g -> f t,
  deltap (in2p t) f g -> g t,
  mup (nup t) -> t,
  ecp (icp f) t -> f t,

  pi1n (pairn t1 t2) -> t1,
  pi2n (pairn t1 t2) -> t2,
  deltan (in1n t) f g -> f t,
  deltan (in2n t) f g -> g t,
  mun (nun t) -> t,
  ecn (icn f) t -> f t,

  abs (pairp t s) (in1n u) -> abs (ecp t u) (ecn u t),
  abs (pairp t s) (in2n u) -> abs (ecp s u) (ecn u s),
  abs (in1p u) (pairn t s) -> abs (ecp u t) (ecn t u),
  abs (in2p u) (pairn t s) -> abs (ecp u s) (ecn s u),
  abs (nup t) (nun s) -> abs (ecp s t) (ecn t s),
)
```

3.1 ... its extension for $\lambda_\eta^{\mathbf{prk}}$

This formalization can be extended to $\lambda_\eta^{\text{PRK}}$, simply by adding the corresponding rules.

```
(FUN
  pi1p : a -> a
  pi2p : a -> a
  pairp : a -> a -> a
  in1p  : a -> a
  in2p  : a -> a
  deltap : a -> (a -> a) -> (a -> a) -> a
  nup : a -> a
  mup : a -> a
  icp : (a -> a) -> a
  ecp : a -> a -> a

  pi1n : a -> a
  pi2n : a -> a
  pairn : a -> a -> a
  in1n  : a -> a
  in2n  : a -> a
  deltan : a -> (a -> a) -> (a -> a) -> a
  nun : a -> a
  mun : a -> a
  icn : (a -> a) -> a
  ecn : a -> a -> a

  abs : a -> a -> a
)

(VAR
  f : a -> a
  g : a -> a
  t : a
  s : a
  u : a
  t1 : a
  t2 : a
  x : a
)

(RULES
  pi1p (pairp t1 t2) -> t1,
  pi2p (pairp t1 t2) -> t2,
  deltap (in1p t) f g -> f t,
  deltap (in2p t) f g -> g t,
  mup (nup t) -> t,
  ecp (icp f) t -> f t,
```

```
  pi1n (pairn t1 t2) -> t1,
  pi2n (pairn t1 t2) -> t2,
  deltan (in1n t) f g -> f t,
  deltan (in2n t) f g -> g t,
  mun (nun t) -> t,
  ecn (icn f) t -> f t,

  abs (pairp t s) (in1n u) -> abs (ecp t u) (ecn u t),
  abs (pairp t s) (in2n u) -> abs (ecp s u) (ecn u s),
  abs (in1p u) (pairn t s) -> abs (ecp u t) (ecn t u),
  abs (in2p u) (pairn t s) -> abs (ecp u s) (ecn s u),
  abs (nup t) (nun s) -> abs (ecp s t) (ecn t s),

  icp (\x. ecp t x) -> t,
  icn (\x. ecn t x) -> t,
)
```

3.2 ... and its extension for $\lambda_2^{\mathbf{prk}}$

Finally, it can also be extended to the second order version of our calculus, λ_2^{PRK}, by introducing the necessary term constructors and their corresponding reduction rules.

```
(FUN
  pi1p : a -> a
  pi2p : a -> a
  pairp : a -> a -> a
  in1p  : a -> a
  in2p  : a -> a
  deltap : a -> (a -> a) -> (a -> a) -> a
  nup : a -> a
  mup : a -> a
  icp : (a -> a) -> a
  ecp : a -> a -> a
  lamp : (T -> a) -> a
  appp : a -> T -> a
  exp : T -> a -> a
  dexp : (T -> a -> a) -> a -> a

  pi1n : a -> a
  pi2n : a -> a
  pairn : a -> a -> a
  in1n  : a -> a
  in2n  : a -> a
  deltan : a -> (a -> a) -> (a -> a) -> a
  nun : a -> a
  mun : a -> a
  icn : (a -> a) -> a
```

```
  ecn : a -> a -> a
  lamn : (T -> a) -> a
  appn : a -> T -> a
  exn : T -> a -> a
  dexn : (T -> a -> a) -> a -> a

  abs : a -> a -> a
)

(VAR
  f : a -> a
  g : a -> a
  t : a
  s : a
  u : a
  t1 : a
  t2 : a

  A : T
  B : T
  Flam : T -> a
  Fdex : T -> a -> a
)

(RULES
  pi1p (pairp t1 t2) -> t1,
  pi2p (pairp t1 t2) -> t2,
  deltap (in1p t) f g -> f t,
  deltap (in2p t) f g -> g t,
  mup (nup t) -> t,
  ecp (icp f) t -> f t,
  appp (lamp Flam) A -> Flam A,
  dexp Fdex (exp A t) -> Fdex A t,

  pi1n (pairn t1 t2) -> t1,
  pi2n (pairn t1 t2) -> t2,
  deltan (in1n t) f g -> f t,
  deltan (in2n t) f g -> g t,
  mun (nun t) -> t,
  ecn (icn f) t -> f t,
  appn (lamn Flam) A -> Flam A,
  dexn Fdex (exn A t) -> Fdex A t,

  abs (pairp t s) (in1n u) -> abs (ecp t u) (ecn u t),
  abs (pairp t s) (in2n u) -> abs (ecp s u) (ecn u s),
  abs (in1p u) (pairn t s) -> abs (ecp u t) (ecn t u),
  abs (in2p u) (pairn t s) -> abs (ecp u s) (ecn s u),
```

```
  abs (nup t) (nun s) -> abs (ecp s t) (ecn t s),
  abs (lamp Flam) (exn A t) -> abs (ecp (Flam A) t) (ecn t (Flam A)),
  abs (exp A t) (lamn Flam) -> abs (ecp t (Flam A)) (ecn (Flam A) t),
)
```

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