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The Vectorial Lambda Calculus Revisited

Tesis de Licenciatura en Ciencias de la Computación

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A mi familia,
en especial mis abuelos y mi hermano.

Abstract

We revisit the Vectorial λ-Calculus [3], a system that provides a way to model a vector space of terms by extending the classic terms of the λ-calculus with linear combinations of them; and by introducing a type system on top. The system can be summarized by the slogan

If $\Gamma \vdash \mathbf{t} : T$ and $\Gamma \vdash \mathbf{r} : R$ then $\Gamma \vdash \alpha \cdot \mathbf{t} + \beta \cdot \mathbf{r} : \alpha \cdot T + \beta \cdot R$.

However, the type system in Vectorial only provides a weakened version of the Subject Reduction property. We prove that our revised Vectorial λ-Calculus supports the standard version of said property as well as many others in the original system, such as Progress. We also introduce the concept of weight of types and terms, and a new property relating the weight of a term with the weight of its type.

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Contents

Chapter 1

Introduction

The study of quantum computing has implications on the way we understand physics today, and can be considered a new computational paradigm altogether, which also has implications on the way we understand computers, algorithms and logic. This field of study then proposes that a computer is in fact a quantum physical system and, as such, it behaves according to the laws of quantum physics. Modeling computers this way (and more specifically, computer algorithms), has yielded important algorithmic results in the last few decades [8, 12, 17], which have no classical counterpart; to such a degree that those algorithms are written using the primitive model of quantum circuits, akin to how the algorithms of classic computation were modeled back in the 1950s: there was no higher-level model available to write them. Consequently, several researchers have started to develop new programming languages that capture the essence of the quantum systems, while providing a higher-level abstraction. Selinger [16] established the "quantum data, classic control" paradigm, which proposed that quantum computers will run in specialized devices attached to classical computers, with the latter instructing the former which operations to perform over which qubits. In this scheme, the classical computer is the one that performs the measurements on the qubits to retrieve the classical bits to continue running the program. Another approach is to model the programming language following a "quantum data, quantum control" paradigm [1, 5], which avoids any additional classical registers and programming structures. Recent results suggest that quantum control may turn out to be more efficient than classical control when dealing with black-box algorithms [7, 15].

1.1 Linear-algebraic λ-Calculus

Following the quantum control paradigm, the untyped linear-algebraic λ-calculus *Lineal* (λ_{lin}) [5] was introduced. λ_{lin} is a minimalistic language that is able to model high-level computation with linear algebra, and therefore provides a computational definition of vector spaces and bilinear functions. The first problem addressed by this language was how to model higher-order computable operators over infinite dimensional vector spaces, which serves as a basis for studying wider notions of computability upon abstract vector spaces, whatever the interpretation of the vectors might be (probabilities, number of computational paths leading to one result, etc.). Therefore, the system is able to represent linear combination of terms, which is key to model one of the most important characteristics of quantum systems: the state vector of a system. Thus, the terms are modeled as said state vectors, and if $\mathbf{t}$ and $\mathbf{u}$ are valid terms, then it is possible to write

$$\alpha \cdot \mathbf{t} + \beta \cdot \mathbf{u}$$

which is also a valid term that represents the superposition of the state vectors $\mathbf{t}$ and $\mathbf{u}$. However, the downside of this generality in the context of quantum computing, is that the operators are not restricted to being unitary (as required by quantum physics), but it serves as a starting point to build more specialized quantum languages by enforcing such restrictions, following for example [9].

1.2 The Vectorial λ-Calculus

While λ_{lin} is enough to model the axioms of vector spaces in the simplest and most general form, it remains an untyped λ-calculus. Consider then the following example, $\mathbf{Y_b} = (\lambda x.(\mathbf{b} + (x)\ x))\ \lambda x.(\mathbf{b} + (x)\ x)$. Then $\mathbf{Y_b}$ is reduced to $\mathbf{b} + \mathbf{Y_b}$. So the term $\mathbf{Y_b} - \mathbf{Y_b}$ is reduced to $\mathbf{0}$, but also is reduced to $\mathbf{b} + \mathbf{Y_b} - \mathbf{Y_b}$ and hence to $\mathbf{b}$, breaking confluence. To solve this, λ_{lin} introduced some restrictions to the reduction rules such as restricting the reduction rule $\alpha \cdot \mathbf{t} + \beta \cdot \mathbf{t} \to (\alpha + \beta) \cdot \mathbf{t}$ to be applied only when $\mathbf{t}$ is a closed normal term. This rule forbids the reduction $\mathbf{Y_b} - \mathbf{Y_b} \to \mathbf{0}$, thus regaining confluence. However, even then the system still admits terms that do not normalise, such as the looping term $\Omega = (\lambda x.(x)\ x)\ (\lambda x.(x)\ x)$.

The *Vectorial λ-calculus* [3], denoted as λ^{vec}, restricted λ_{lin} by providiing a formal account of linear operators and vectors at the level of the type system, including both scalars and sums of types, which can then be summarized by the slogan:

If $\Gamma \vdash \mathbf{t} : T$ and $\Gamma \vdash \mathbf{r} : R$ then $\Gamma \vdash \alpha \cdot \mathbf{t} + \beta \cdot \mathbf{r} : \alpha \cdot T + \beta \cdot R$.

The type system exhibits the following accomplishments:

- The typed language features a weakened Subject Reduction property.
- The typed language features strong normalisation.
- In general, if $\mathbf{t}$ has type $\sum_i \alpha_i \cdot U_i$, then it must be reduced to a $\mathbf{t}'$ of the form $\sum_{ij} \beta_{ij} \cdot \mathbf{b}_{ij}$, where: the $\mathbf{b}_{ij}$'s are basis terms of unit type U_i, and $\sum_{ij} \beta_{ij} = \alpha_i$.
- In particular, finite vectors, matrices and tensorial products can be encoded within λ^{vec}. In this case, the type of the encoded expressions coincides with the result of the expression.

Notice that since the type system features strong normalisation, the terms $\mathbf{Y_b} - \mathbf{Y_b}$ and Ω are no longer an issue, since they are not well-typed, thus allowing us to remove many of the restrictions and consider a more canonical set of rewritten rules [2, 3, 6, 10].

1.3 Thesis plan

The main focus of this work is to bring back the property of Subject Reduction, while preserving as many properties of the original system as possible. We also introduce the concept of weight of terms and types, which represents the sum of all the components of the vectors being modeled by them.

Chapter 2: In this chapter we study λ^{vec} by presenting an in-depth analysis of the design choices behind the system. We also discuss some of its limitations, particularly the weakened version of the Subject Reduction property, whose restoration to the standard version of the property is the main focus of this work.

Chapter 3: We present a revised version of λ^{vec}, and discuss the design decisions behind the revision in order to regain the standard version of the Subject Reduction property.

Chapter 4: Here we prove that our revised version of the λ^{vec} satisfies the standard version of the Subject Reduction property, and we present and prove all the lemmas and definitions needed.
Chapter 5: In this chapter we present proof for other desirable properties of the system:

- Progress (Section 5.1).
- Weight Preservation (Section 5.2): We prove that once a term is reduced, the resulting value has the same weight as its type.

Chapter 6: Finally, we summarize the accomplishments, present several examples of vector encoding using the new system and show some indications for future work.

Chapter 2

The Vectorial λ-Calculus

Chapter Summary

We examine the terms and types that constitute λ^{vec}, and we show how the system is able to characterise vectors, using the Hadamard Quantum Gate as an example of how matrices can be encoded in the system.
We also show two system properties: Strong Normalisation of terms, and λ^{vec}'s weakened Subject Reduction, further examining the latter and presenting the reasons why the standard formulation of the property is not directly satisfied. ■

We formally present λ^{vec} [3], and analyse the different design choices behind it. λ^{vec} is based on λ_{lin} [5], which admits the classical constructs of λ-calculi: variables $x, y, \ldots$, λ-abstractions $\lambda x.\mathbf{s}$, and application $(\mathbf{s})\ \mathbf{t}$. It also admits linear combinations of terms: $\mathbf{0}$, $\mathbf{s} + \mathbf{t}$ and $\alpha \cdot \mathbf{s}$ are terms, where the scalar α ranges over a ring. As in λ_{lin}, it follows a call-by-basis strategy, in the sense that $(\lambda x.\mathbf{r})\ (\mathbf{s} + \mathbf{t})$ is first reduced to $(\lambda x.\mathbf{r})\ \mathbf{s} + (\lambda x.\mathbf{r})\ \mathbf{t}$ until *basis terms* (i.e. values in the standard sense) are reached, at which point beta-reduction applies.

The set of normal forms of terms can then be interpreted as a module and the term $(\lambda x.\mathbf{r})\ \mathbf{s}$ can be seen as the application of the linear operator $\lambda x.\mathbf{r}$ to the vector $\mathbf{s}$.

λ^{vec} then extends λ_{lin} by providing a formal account of linear operators and vectors at the level of the type system.

Chapter plan. In Section 2.1 we examine the terms and the reduction rules. In Section 2.2 we present the type system along with the typing rules, and show some examples of how vectorial computations can be encoded with it. Finally, in Section 2.3 we show the system's properties and we study the weakened version of the Subject Reduction property that λ^{vec} proposes, and the reason why its standard formulation cannot be directly satisfied.

2.1 The terms

We begin by considering the untyped language of λ^{vec} as described in Figure 2.1. For the function application, we use the Krivine's notation [14]: The term $(\mathbf{s})\ \mathbf{t}$ passes the argument $\mathbf{t}$ to the function $\mathbf{s}$.

The terms are divided in two categories:

- Basis terms: The only terms that can be used in a β-reduction step.
- General terms.

This design, as in [2,5,10], follows a call-by-basis reduction strategy: only the outermost redexes are reduced: a redex is reduced only when its right hand side has been reduced to a basis term (variable or lambda abstraction), cf. Group B in Figure 2.1.

Besides the β-reduction rule, the system presents other reduction rules which follow an oriented version of the axioms of vector spaces, determining the behaviour of sums and products. As such, they are divided into the following groups: Elementary (E), Factorisation (F), Application (A) and the Beta Reduction (B).

Essentially, the E and F groups rules, presented in [4], consist in capturing the equations of vector spaces in an oriented rewrite system. For example, $0 \cdot \mathbf{s}$ is reduced to $\mathbf{0}$, as $0 \cdot \mathbf{s} = \mathbf{0}$ is valid in vector spaces.

It should be noted that this set of algebraic rules is locally confluent [3], and does not introduce loops. In particular, the two rules stating $\alpha\cdot(\mathbf{t}+\mathbf{r}) \to \alpha\cdot\mathbf{t}+\alpha\cdot\mathbf{r}$ and $\alpha\cdot\mathbf{t}+\beta\cdot\mathbf{t} \to (\alpha+\beta)\cdot\mathbf{t}$ are not the inverse of the other when $\mathbf{r} = \mathbf{t}$. Indeed,

$$\alpha \cdot (\mathbf{t} + \mathbf{t}) \to \alpha \cdot \mathbf{t} + \alpha \cdot \mathbf{t} \to (\alpha + \alpha) \cdot \mathbf{t}$$

but not the other way around.

The Group A of rules formalises the fact that a general term $\mathbf{t}$ is thought of as a linear combination of terms $\alpha \cdot \mathbf{r} + \beta \cdot \mathbf{r}'$ and the fact that the application is distributive on the left *and* on the right. When $\mathbf{s}$ is applied to such a superposition, $(\mathbf{s})\ \mathbf{t}$ is reduced to $\alpha \cdot (\mathbf{s})\ \mathbf{r} + \beta \cdot (\mathbf{s})\ \mathbf{r}'$. The term $\mathbf{0}$ is the empty linear combination of terms, explaining the last two rules of Group A.

Terms are considered modulo associativity and commutativity of the operator $+$, making the reduction into an *AC-rewrite system* [13].

Scalars (notation $\alpha, \beta, \gamma, \dots$) form a ring $(\mathsf{S}, +, \times)$, where the scalar 0 is the unit of addition and 1 the unit of multiplication. We use the shortcut notation $\mathbf{s} - \mathbf{t}$ instead of $\mathbf{s} + (-1) \cdot \mathbf{t}$.

The set of free variables of a term is defined as usual: the only operator binding variables is the λ-abstraction. The operation of substitution on terms (notation $\mathbf{t}[\mathbf{b}/x]$) is also defined as usual, by taking care of variable renaming to avoid capture.

For a linear combination, the substitution is defined as follows:

$$(\alpha \cdot \mathbf{t} + \beta \cdot \mathbf{r})[\mathbf{b}/x] = \alpha \cdot \mathbf{t}[\mathbf{b}/x] + \beta \cdot \mathbf{r}[\mathbf{b}/x]$$

$$\begin{array}{rrl} \textit{Terms:} & \mathbf{r},\mathbf{s},\mathbf{t},\mathbf{u} ::= & \mathbf{b} \mid (\mathbf{t})\ \mathbf{r} \mid \mathbf{0} \mid \alpha \cdot \mathbf{t} \mid \mathbf{t} + \mathbf{r} \\ \textit{Basis terms:} & \mathbf{b} ::= & x \mid \lambda x.\mathbf{t} \end{array}$$

Group E:

$$\begin{array}{l} 0 \cdot \mathbf{t} \to \mathbf{0} \\ 1 \cdot \mathbf{t} \to \mathbf{t} \\ \alpha \cdot \mathbf{0} \to \mathbf{0} \\ \alpha \cdot (\beta \cdot \mathbf{t}) \to (\alpha \times \beta) \cdot \mathbf{t} \\ \alpha \cdot (\mathbf{t} + \mathbf{r}) \to \alpha \cdot \mathbf{t} + \alpha \cdot \mathbf{r} \end{array}$$

Group F:

$$\begin{array}{l} \alpha \cdot \mathbf{t} + \beta \cdot \mathbf{t} \to (\alpha + \beta) \cdot \mathbf{t} \\ \alpha \cdot \mathbf{t} + \mathbf{t} \to (\alpha + 1) \cdot \mathbf{t} \\ \mathbf{t} + \mathbf{t} \to (1 + 1) \cdot \mathbf{t} \\ \mathbf{t} + \mathbf{0} \to \mathbf{t} \end{array}$$

Group B:

$$(\lambda x.\mathbf{t})\ \mathbf{b} \to \mathbf{t}[\mathbf{b}/x]$$

Group A:

$$\begin{array}{l} (\mathbf{t} + \mathbf{r})\ \mathbf{u} \to (\mathbf{t})\ \mathbf{u} + (\mathbf{r})\ \mathbf{u} \\ (\mathbf{t})\ (\mathbf{r} + \mathbf{u}) \to (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{u} \\ (\alpha \cdot \mathbf{t})\ \mathbf{r} \to \alpha \cdot (\mathbf{t})\ \mathbf{r} \\ (\mathbf{t})\ (\alpha \cdot \mathbf{r}) \to \alpha \cdot (\mathbf{t})\ \mathbf{r} \\ (\mathbf{0})\ \mathbf{t} \to \mathbf{0} \\ (\mathbf{t})\ \mathbf{0} \to \mathbf{0} \end{array}$$

$$\frac{\mathbf{t} \to \mathbf{r}}{\alpha \cdot \mathbf{t} \to \alpha \cdot \mathbf{r}} \qquad \frac{\mathbf{t} \to \mathbf{r}}{\mathbf{u} + \mathbf{t} \to \mathbf{u} + \mathbf{r}} \qquad \frac{\mathbf{t} \to \mathbf{r}}{(\mathbf{u})\ \mathbf{t} \to (\mathbf{u})\ \mathbf{r}} \qquad \frac{\mathbf{t} \to \mathbf{r}}{(\mathbf{t})\ \mathbf{u} \to (\mathbf{r})\ \mathbf{u}} \qquad \frac{\mathbf{t} \to \mathbf{r}}{\lambda x.\mathbf{t} \to \lambda x.\mathbf{r}}$$

Figure 2.1: Syntax, reduction rules and context rules of λ^{vec}.

Booleans in the Vectorial λ-Calculus

Both in λ^{vec} and in quantum computing we can interpret the notion of booleans. In the former, we can consider the usual boolean terms $\mathbf{true} = \lambda x.\lambda y.x$ and $\mathbf{false} = \lambda x.\lambda y.y$ whereas, in the latter, we consider the regular quantum bits $\mathbf{true} = |0\rangle$ and $\mathbf{false} = |1\rangle$.

In λ^{vec}, a representation of *if* $\mathbf{r}$ *then* $\mathbf{s}$ *else* $\mathbf{t}$ needs to take into account the special relation between sums and applications. It is incorrect to directly encode this test as the usual $((\mathbf{r})\ \mathbf{s})\ \mathbf{t}$. Indeed, if $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$ were respectively the terms $\mathbf{true}$, $\mathbf{s}_1+\mathbf{s}_2$ and $\mathbf{t}_1+\mathbf{t}_2$, the term $((\mathbf{true})\ (\mathbf{s}_1+\mathbf{s}_2))\ (\mathbf{t}_1+\mathbf{t}_2)$ would be reduced to $((\mathbf{true})\ \mathbf{s}_1)\ \mathbf{t}_1+((\mathbf{true})\ \mathbf{s}_1)\ \mathbf{t}_2+((\mathbf{true})\ \mathbf{s}_2)\ \mathbf{t}_1+((\mathbf{true})\ \mathbf{s}_2)\ \mathbf{t}_2$, then to $2\cdot\mathbf{s}_1+2\cdot\mathbf{s}_2$ instead of $\mathbf{s}_1+\mathbf{s}_2$.

In order to "freeze" the computations in each branch of the test so that the sum does not distribute over the application, λ^{vec} uses *thunks* [5]: they encode the test as $\{((\mathbf{r})\ \ [\mathbf{s}])\ \ [\mathbf{t}]\}$, where $[-]$ is the term $\lambda f.-$ with f a fresh, unused term variable, and where $\{-\}$ is the term $(-)\ \lambda x.x$. Then,

$$
\begin{aligned}
\{((\mathbf{r})\ \ [\mathbf{s}])\ \ [\mathbf{t}]\} &= (((\mathbf{r})\ (\lambda f.\mathbf{s}))\ (\lambda g.\mathbf{t}))\ (\lambda x.x)\\
&= (((\lambda x.\lambda y.x)\ (\lambda f.\mathbf{s}_1+\mathbf{s}_2))\ (\lambda g.\mathbf{t}_1+\mathbf{t}_2))\ (\lambda x.x)\\
&\to ((\lambda y.\lambda f.\mathbf{s}_1+\mathbf{s}_2)\ (\lambda g.\mathbf{t}_1+\mathbf{t}_2))\ (\lambda x.x)\\
&\to (\lambda f.\mathbf{s}_1+\mathbf{s}_2)\ (\lambda x.x)\\
&\to \mathbf{s}_1+\mathbf{s}_2
\end{aligned}
$$

The $[-]$ encoding "freezes" the linearity while the $\{-\}$ encoding "releases" it. Then the term *if* $\mathbf{true}$ *then* $(\mathbf{s}_1+\mathbf{s}_2)$ *else* $(\mathbf{t}_1+\mathbf{t}_2)$ is reduced to the term $\mathbf{s}_1+\mathbf{s}_2$ as could be expected. Note that this test is linear, in the sense that the term *if* $(\alpha\cdot\mathbf{true}+\beta\cdot\mathbf{false})$ *then* $\mathbf{s}$ *else* $\mathbf{t}$ is reduced to $\alpha\cdot\mathbf{s}+\beta\cdot\mathbf{t}$, implementing the so-called *quantum-if* [1].

Quantum computing deals with complex, linear combinations of terms, and a typical computation is run by applying linear unitary operations on the terms, called *gates*. For example, the Hadamard gate $\mathbf{H}$ acts on the space of booleans spanned by $\mathbf{true}$ and $\mathbf{false}$. It sends $\mathbf{true}$ to $\frac{1}{\sqrt{2}}(\mathbf{true}+\mathbf{false})$ and $\mathbf{false}$ to $\frac{1}{\sqrt{2}}(\mathbf{true}-\mathbf{false})$. If x is a quantum bit, the value $(\mathbf{H})\ x$ can be represented as the quantum test

$$
(\mathbf{H})\ x \quad := \quad \mathit{if}\ x\ \mathit{then}\ \left(\frac{1}{\sqrt{2}}(\mathbf{true}+\mathbf{false})\right)\ \mathit{else}\ \left(\frac{1}{\sqrt{2}}(\mathbf{true}-\mathbf{false})\right).
$$

As developed in [5], we can simulate this operation in λ^{vec} using the test construction we just described:

$$
\{(\mathbf{H})\ x\} \quad := \quad \left\{\left((x)\ \left[\frac{1}{\sqrt{2}}\cdot\mathbf{true}+\frac{1}{\sqrt{2}}\cdot\mathbf{false}\right]\right)\ \left[\frac{1}{\sqrt{2}}\cdot\mathbf{true}-\frac{1}{\sqrt{2}}\cdot\mathbf{false}\right]\right\}.
$$

Note that the thunks are necessary: without thunks the term

$$
\left((x)\ \left(\frac{1}{\sqrt{2}}\cdot\mathbf{true}+\frac{1}{\sqrt{2}}\cdot\mathbf{false}\right)\right)\ \left(\frac{1}{\sqrt{2}}\cdot\mathbf{true}-\frac{1}{\sqrt{2}}\cdot\mathbf{false}\right)
$$

would be reduced to the term

$$
\frac{1}{2}(((x)\ \mathbf{true})\ \mathbf{true}+((x)\ \mathbf{true})\ \mathbf{false}+((x)\ \mathbf{false})\ \mathbf{true}+((x)\ \mathbf{false})\ \mathbf{false}),
$$

which is fundamentally different from the term $\mathbf{H}$ we are trying to emulate. With the same procedure, λ^{vec} can "encode" any matrix. If the space is of some general dimension n, instead of the basis elements $\mathbf{true}$ and $\mathbf{false}$, the terms $\lambda x_1.\cdots.\lambda x_n.x_i$ can choosen for $i=\{1,\ldots,n\}$ to encode the base of the space. We can also take tensor products of qubits. More details of these encodings are provided in Appendix A.1.

2.2 The type system

2.2.1 Intuitions

Before diving into the technicalities of the definition, we discuss the rationale behind the construction of the type-system.

Superposition of types

λ^{vec} begins by incorporating the notion of scalars into the type system: If A is a valid type, the construction $\alpha \cdot A$ is also a valid type and if the terms $\mathbf{s}$ and $\mathbf{t}$ are of type A, the term $\alpha \cdot \mathbf{s} + \beta \cdot \mathbf{t}$ is of type $(\alpha + \beta) \cdot A$. This was achieved in [2] and it allows to distinguish between the functions $\lambda x.(1 \cdot x)$ and $\lambda x.(2 \cdot x)$: the former is of type $A \to A$ whereas the latter is of type $A \to (2 \cdot A)$.

The terms **true** and **false** can be typed in the usual way with $\mathcal{B} = X \to (X \to X)$, for a fixed type X. So let us consider the term $\frac{1}{\sqrt{2}} \cdot (\mathbf{true} - \mathbf{false})$. Using the above addition to the type system, this term should be of type $0 \cdot \mathcal{B}$, a type which fails to exhibit the fact that we have a superposition of terms.

To address this problem, λ^{vec} admits sums of types. For instance, provided that $\mathcal{T} = X \to (Y \to X)$ and $\mathcal{F} = X \to (Y \to Y)$, we can type the term $\frac{1}{\sqrt{2}} \cdot (\mathbf{true} - \mathbf{false})$ with $\frac{1}{\sqrt{2}} \cdot (\mathcal{T} - \mathcal{F})$, which has L_2-norm 1, just like the type of **false** has norm one.

At this stage the type system is able to type the term

$$\mathbf{H} = \lambda x.\left\{\left((x)\ \left[\frac{1}{\sqrt{2}} \cdot \mathbf{true} + \frac{1}{\sqrt{2}} \cdot \mathbf{false}\right]\right)\ \left[\frac{1}{\sqrt{2}} \cdot \mathbf{true} - \frac{1}{\sqrt{2}} \cdot \mathbf{false}\right]\right\}$$

Indeed, as previously mentioned, the thunk construction $[-]$ is simply $\lambda f.(-)$ where f is a fresh variable and that $\{-\}$ is $(-)\ \lambda x.x$. So whenever $\mathbf{t}$ has type A, $[\mathbf{t}]$ has type $\mathbf{I} \to A$ with $\mathbf{I}$ an identity type of the form $Z \to Z$, and $\{\mathbf{t}\}$ has type A whenever $\mathbf{t}$ has type $\mathbf{I} \to A$. The term $\mathbf{H}$ can then be typed with $\left(\left(\mathbf{I} \to \frac{1}{\sqrt{2}}.(\mathcal{T} + \mathcal{F})\right) \to \left(\mathbf{I} \to \frac{1}{\sqrt{2}}.(\mathcal{T} - \mathcal{F})\right) \to \mathbf{I} \to T\right) \to T$, with T any fixed type.

Let us now try to type the term $(\mathbf{H})\ \mathbf{true}$. This is made possible by taking T to be $\frac{1}{\sqrt{2}} \cdot (\mathcal{T} + \mathcal{F})$. But then, if we want to type the term $(\mathbf{H})\ \mathbf{false}$, T needs to be equal to $\frac{1}{\sqrt{2}} \cdot (\mathcal{T} - \mathcal{F})$. It follows that we cannot type the term $(\mathbf{H})\ \left(\frac{1}{\sqrt{2}} \cdot \mathbf{true} + \frac{1}{\sqrt{2}} \cdot \mathbf{false}\right)$ since it is not possible to conciliate the two constraints on T.

To address this problem, λ^{vec} introduces the universal abstraction in the type system, making it *à la System F*. The term $\mathbf{H}$ can now be typed with $\forall T.((\mathbf{I} \to \frac{1}{\sqrt{2}} \cdot (\mathcal{T} + \mathcal{F})) \to (\mathbf{I} \to \frac{1}{\sqrt{2}} \cdot (\mathcal{T} - \mathcal{F})) \to \mathbf{I} \to T) \to T$ and the types $\mathcal{T}$ and $\mathcal{F}$ are updated to be respectively $\forall XY.X \to (Y \to X)$ and $\forall XY.X \to (Y \to Y)$. The terms $(\mathbf{H})\ \mathbf{true}$ and $(\mathbf{H})\ \mathbf{false}$ can both be well-typed with respective types $\frac{1}{\sqrt{2}} \cdot (\mathcal{T} + \mathcal{F})$ and $\frac{1}{\sqrt{2}} \cdot (\mathcal{T} - \mathcal{F})$, as expected.

Type variables, units and general types

Because of the call-by-basis strategy, variables must range over types that are not linear combination of other types, i.e. *unit types*. To illustrate this necessity, consider the following counterexample. Suppose λ^{vec} allows variables to have scaled types, such as $\alpha \cdot U$. Then the term $\lambda x.x + y$ could have type $(\alpha \cdot U) \to \alpha \cdot U + V$ (with y of type V). Let $\mathbf{b}$ be of type U, then $(\lambda x.x + y)\ (\alpha \cdot \mathbf{b})$ has type $\alpha \cdot U + V$, however

$$(\lambda x.x + y)\ (\alpha \cdot \mathbf{b}) \to \alpha \cdot (\lambda x.x + y)\ \mathbf{b} \to \alpha \cdot (\mathbf{b} + y) \to \alpha \cdot \mathbf{b} + \alpha \cdot y\ ,$$

which is problematic since the type $\alpha \cdot U + V$ does not reflect such a superposition. Hence, the left side of an arrow will be required to be a unit type; that is, not a superposition of types. This is achieved by the grammar defined in Figure 2.2.

Type variables, however, do not always have to be unit types. In fact, a universal abstraction of a general type was needed in the previous section in order to type the term **H**. Therefore, to distinguish a general type variable from a unit type variable (to make sure that only unit types appear at the left of arrows), λ^{vec} defines two sorts of type variables: the variables $\mathcal{X}$ to be replaced with unit types, and $\mathbb{X}$ to be replaced with any type (we use X when we mean either one). The type $\mathcal{X}$ is a unit type whereas the type $\mathbb{X}$ is not.

In particular, the type of **true**, $\mathcal{T}$, is now $\forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{X}$, the type of **false**, $\mathcal{F}$, is $\forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{Y}$ and the type of **H** is

$$\forall \mathbb{X}. \left(\left(\mathbf{I} \to \frac{1}{\sqrt{2}} \cdot (\mathcal{T} + \mathcal{F}) \right) \to \left(\mathbf{I} \to \frac{1}{\sqrt{2}} \cdot (\mathcal{T} - \mathcal{F}) \right) \to \mathbf{I} \to \mathbb{X} \right) \to \mathbb{X}.$$

Note how the left sides of all arrows remain unit types.

The term 0

The term $\mathbf{0}$ will naturally have the type $0 \cdot T$, for any inhabited type T (enforcing the intuition that the term $\mathbf{0}$ is essentially a normal form of programs of the form $\mathbf{t} - \mathbf{t}$).

Even though there are reduction rules such as $\mathbf{t} + \mathbf{0} \to \mathbf{t}$, λ^{vec} does not allow the equivalence $T + 0 \cdot R \equiv T$. To understand why that decision was made, consider the following example. Let $\lambda x.x$ be of type $U \to U$ and let $\mathbf{r}$ be of type R. The term $\lambda x.x + \mathbf{r} - \mathbf{r}$ is of type $(U \to U) + 0 \cdot R$, and if λ^{vec} allowed such equivalence, $(U \to U)$. If we were to choose $\mathbf{b}$ of type U, we would be allowed to say that $(\lambda x.x + \mathbf{r} - \mathbf{r})\ \mathbf{b}$ is of type U. This term is reduced to $\mathbf{b} + (\mathbf{r})\ \mathbf{b} - (\mathbf{r})\ \mathbf{b}$. But if the type system is reasonable enough, it should at least be able to type $(\mathbf{r})\ \mathbf{b}$. However, since there are no constraints on the type R, this is difficult to enforce.

The problem comes from the fact that along the typing of $\mathbf{r} - \mathbf{r}$, the type of $\mathbf{r}$ is lost in the equivalence $(U \to U) + 0 \cdot R \equiv U \to U$. Therefore, while the terms form a module (semantically), the types form a weak module in the sense of [10].

2.2.2 Formalisation

We now present the type system of λ^{vec}: we first describe the language of types, then present the typing rules.

Definition of types

Types are defined in Figure 2.2 (top). As with the terms, there are two kinds: *unit types* and general types, that is, linear combinations of types. Unit types include all types of *System F* [11, Ch. 11] and intuitively they are used to type basis terms. The arrow type admits only a unit type in its domain. This is due to the fact that the argument of a λ-abstraction can only be substituted by a basis term, as discussed in Section 2.2.1.

As previously mentioned, the type system features two sorts of variables: unit variables $\mathcal{X}$ and general variables $\mathbb{X}$. The former can only be substituted by a unit type whereas the latter can be substituted by any type. We use the notation X when the type variable is unspecified. The substitution of $\mathcal{X}$ by U (resp. $\mathbb{X}$ by S) in T is defined as usual and is written $T[U/\mathcal{X}]$ (resp. $T[S/\mathbb{X}]$). We use the notation $T[A/X]$ to say: "if X is a unit variable, then A is a unit type, otherwise A is a general type".

$$
\begin{array}{llll}
\textit{Types:} & T, R, S & ::= & U \mid \alpha \cdot T \mid T + R \mid \mathbb{X} \\
\textit{Unit types:} & U, V, W & ::= & \mathcal{X} \mid U \to T \mid \forall \mathcal{X}.U \mid \forall \mathbb{X}.U
\end{array}
$$

$$
\begin{array}{rclrcl}
1 \cdot T & \equiv & T & \alpha \cdot T + \beta \cdot T & \equiv & (\alpha + \beta) \cdot T \\
\alpha \cdot (\beta \cdot T) & \equiv & (\alpha \times \beta) \cdot T & T + R & \equiv & R + T \\
\alpha \cdot T + \alpha \cdot R & \equiv & \alpha \cdot (T + R) & T + (R + S) & \equiv & (T + R) + S
\end{array}
$$

$$
\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \mathbf{0} : 0 \cdot T}\ 0_I \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T}\ \to_I
$$

$$
\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i) \qquad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j / \vec{X}]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j / \vec{X}]}\ \to_E
$$

$$
\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i \quad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot \forall X.U_i}\ \forall_I \qquad \frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot \forall X.U_i}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i[A/X]}\ \forall_E
$$

$$
\frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha \cdot \mathbf{t} : \alpha \cdot T}\ \alpha_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R}\ +_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad T \equiv R}{\Gamma \vdash \mathbf{t} : R}\ \equiv
$$

Figure 2.2: Types and typing rules of λ^{vec}. We use X when we do not want to specify if it is $\mathcal{X}$ or $\mathbb{X}$, that is, unit variables or general variables respectively. In $T[A/X]$, if $X = \mathcal{X}$, then A is a unit type, and if $X = \mathbb{X}$, then A can be any type. We also write $\forall_I$ and $\forall_{\mathbb{I}}$ (resp. $\forall_{\mathcal{E}}$ and $\forall_{\mathbb{E}}$) when we need to specify which kind of variable is being used.

In particular, for a linear combination, the substitution is defined as follows: $(\alpha\cdot T+\beta\cdot R)[A/X] = \alpha \cdot T[A/X] + \beta \cdot R[A/X]$. We also use the vectorial notation $T[\vec{A}/\vec{X}]$ for $T[A_1/X_1]\cdots[A_n/X_n]$ if $\vec{X} = X_1,\dots,X_n$ and $\vec{A} = A_1,\dots,A_n$, and also $\forall\vec{X}$ for $\forall X_1 \dots X_n = \forall X_1.\dots.\forall X_n$.

The equivalence relation $\equiv$ on types is defined as a congruence. Notice that this equivalence makes the types into a weak module over the scalars: they would form a module were it not for the fact that there is no neutral element for the addition. Indeed, the type $0 \cdot T$ is not the neutral element of the addition.

We use the summation ($\sum$) notation without ambiguity, due to the associativity and commutativity equivalences of $+$.

Typing rules

The typing rules are also established in Figure 2.2 (bottom). Contexts are denoted by Γ, Δ, etc, and are defined as sets $\{x : U, \dots\}$, where x is a term variable appearing only once in the set, and U is a unit type. We usually omit the curly braces and just write $x : U, \cdots$.

- The axiom (ax) and the arrow introduction rule $(\to_I)$ are the usual ones.
- The $\forall_I$ and $\forall_E$ allow the introduction and the elimination of universal abstraction for types $(\forall)$, respectively.
- The rule (0_I) to type the term $\mathbf{0}$ takes into account the discussion in Section 2.2.1. This rule ensures that the type of $\mathbf{0}$ is inhabited, discarding problematic types such as $0\cdot\forall X.X$.
- Any sum of typed terms can be typed using rule $(+_I)$.
- Any scaled typed term can be typed with (α_I).
- Rule $(\equiv)$ ensures that equivalent types can be used to type the same terms.
- Finally, the particular form of the arrow-elimination rule $(\to_E)$ is due to the rewrite rules in Group A that distribute sums and scalars over applications. The need and use of this complicated arrow-elimination rule can be illustrated by the following three examples [3].

Example 2.2.1. *Rule $(\to_E)$ is easier to read for trivial linear combinations. It states that provided that $\Gamma \vdash \mathbf{s} : \forall X.U \to S$ and $\Gamma \vdash \mathbf{t} : V$, if there exists some type W such that $V = U[W/X]$ then, since the sequent $\Gamma \vdash \mathbf{s} : V \to S[W/X]$ is valid, we also have $\Gamma \vdash (\mathbf{s})\ \mathbf{t} : S[W/X]$. Hence, the arrow elimination here performs an arrow and a universal abstraction elimination at the same time.*

Example 2.2.2. *Consider the terms $\mathbf{b}_1$ and $\mathbf{b}_2$, of respective types U_1 and U_2. The term $\mathbf{b}_1+\mathbf{b}_2$ is of type $U_1 + U_2$. We would reasonably expect the term $(\lambda x.x)\ (\mathbf{b}_1 + \mathbf{b}_2)$ to also be of type $U_1 + U_2$. This is the case thanks to Rule $(\to_E)$. Indeed, if we type the term $\lambda x.x$ with the type $\forall X.X \to X$ the rule can now be applied. Note that we could not type such a term unless we eliminated the universal abstraction together with the arrow.*

Example 2.2.3. *The projection of a pair of elements is a slightly more involved example. It is possible to encode the notion of pairs and projections in* System F*: $\langle \mathbf{b}, \mathbf{c}\rangle = \lambda x.((x)\ \mathbf{b})\ \mathbf{c}$, $\langle \mathbf{b}', \mathbf{c}'\rangle = \lambda x.((x)\ \mathbf{b}')\ \mathbf{c}'$, $\pi_1 = \lambda x.(x)\ (\lambda y.\lambda z.y)$ and $\pi_2 = \lambda x.(x)\ (\lambda y.\lambda z.z)$. Provided that $\mathbf{b}$, $\mathbf{b}'$, $\mathbf{c}$ and $\mathbf{c}'$ have respective types U, U', V and V', the type of $\langle \mathbf{b}, \mathbf{c}\rangle$ is $\forall X.(U \to V \to X) \to X$ and the type of $\langle \mathbf{b}', \mathbf{c}'\rangle$ is $\forall X.(U' \to V' \to X) \to X$. The term π_1 and π_2 can be typed respectively with $\forall XYZ.((X \to Y \to X) \to Z) \to Z$ and $\forall XYZ.((X \to Y \to Y) \to Z) \to Z$. The term $(\pi_1 + \pi_2)\ (\langle \mathbf{b}, \mathbf{c}\rangle + \langle \mathbf{b}', \mathbf{c}'\rangle)$ is then typable of type $U + U' + V + V'$, thanks to Rule $(\to_E)$. Note that this is consistent with the rewrite system, since it is reduced to $\mathbf{b} + \mathbf{c} + \mathbf{b}' + \mathbf{c}'$.*

2.2.3 Example: Typing Hadamard

In this Section, we formally show how to retrieve the type that was discussed in Section 2.2.1, for the term **H** encoding the Hadamard gate.

Let $\mathbf{true} = \lambda x.\lambda y.x$ and $\mathbf{false} = \lambda x.\lambda y.y$. It is simple to check that

$$\vdash \mathbf{true} : \forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{X} \equiv \mathcal{T},$$
$$\vdash \mathbf{false} : \forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{Y} \equiv \mathcal{F}.$$

We also define the following superpositions:

$$|+\rangle = \frac{1}{\sqrt{2}} \cdot (\mathbf{true} + \mathbf{false}) \qquad \text{and} \qquad |-\rangle = \frac{1}{\sqrt{2}} \cdot (\mathbf{true} - \mathbf{false}).$$

In the same way, we define

$$\boxplus = \frac{1}{\sqrt{2}} \cdot ((\underbrace{\forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{X}}_{\mathcal{T}}) + (\underbrace{\forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{Y}}_{\mathcal{F}})),$$
$$\boxminus = \frac{1}{\sqrt{2}} \cdot ((\underbrace{\forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{X}}_{\mathcal{T}}) - (\underbrace{\forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{Y}}_{\mathcal{F}})).$$

Finally, we recall $[\mathbf{t}] = \lambda x.\mathbf{t}$, where $x \notin FV(\mathbf{t})$ and $\{\mathbf{t}\} = (\mathbf{t})\ I$. So $\{[\mathbf{t}]\} \to \mathbf{t}$. Then it is easy to check that $\vdash [|+\rangle] : I \to \boxplus$ and $\vdash [|-\rangle] : I \to \boxminus$. In order to simplify the notation, let $F = (I \to \boxplus) \to (I \to \boxminus) \to (I \to \mathbb{X})$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : F \vdash x : F}\,ax \qquad x : F \vdash [|+\rangle] : I \to \boxplus}{x : F \vdash (x)\ \ [|+\rangle] : (I \to \boxminus) \to (I \to \mathbb{X})}\to_E \qquad x : F \vdash [|-\rangle] : I \to \boxminus}{x : F \vdash ((x)\ \ [|+\rangle])\ \ [|-\rangle] : I \to \mathbb{X}}\to_E}{x : F \vdash \{((x)\ \ [|+\rangle])\ \ [|-\rangle]\} : \mathbb{X}}\to_E}{\vdash \lambda x.\{((x)\ \ [|+\rangle])\ \ [|-\rangle]\} : F \to \mathbb{X}}\to_I}{\vdash \lambda x.\{((x)\ \ [|+\rangle])\ \ [|-\rangle]\} : \forall \mathbb{X}.((I \to \boxplus) \to (I \to \boxminus) \to (I \to \mathbb{X})) \to \mathbb{X}}\forall_{\mathbb{I}}$$

Now we can apply Hadamard to a qubit and get the right type. Let H be the term $\lambda x.\{((x)\ \ [|+\rangle])\ \ [|-\rangle]\}$.

$$\dfrac{\dfrac{\vdash H : \forall \mathbb{X}.((I \to \boxplus) \to (I \to \boxminus) \to (I \to \mathbb{X})) \to \mathbb{X}}{\vdash H : ((I \to \boxplus) \to (I \to \boxminus) \to (I \to \boxplus)) \to \boxplus}\forall_{\mathbb{E}} \qquad \dfrac{\dfrac{\vdash \mathbf{true} : \forall \mathcal{X}.\forall \mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{X}}{\vdash \mathbf{true} : \forall \mathcal{Y}.(I \to \boxplus) \to \mathcal{Y} \to (I \to \boxplus)}\forall_{\mathcal{E}}}{\vdash \mathbf{true} : (I \to \boxplus) \to (I \to \boxminus) \to (I \to \boxplus)}\forall_{\mathcal{E}}}{\vdash (H)\ \mathbf{true} : \boxplus}\to_E$$

Yet a more interesting example is the following. Let

$$\boxplus_I = \frac{1}{\sqrt{2}} \cdot (((I \to \boxplus) \to (I \to \boxminus) \to (I \to \boxplus)) + ((I \to \boxplus) \to (I \to \boxminus) \to (I \to \boxminus)))$$

That is, $\boxplus$ where the universal abstractions have been instantiated. It is easy to check that $\vdash |+\rangle : \boxplus_I$. Hence,

$$\dfrac{\vdash H : \forall \mathbb{X}.((I \to \boxplus) \to (I \to \boxminus) \to (I \to \mathbb{X})) \to \mathbb{X}}{\vdash (H)\ |+\rangle : \frac{1}{\sqrt{2}} \cdot \boxplus + \frac{1}{\sqrt{2}} \cdot \boxminus}\to_E$$

And since $\frac{1}{\sqrt{2}} \cdot \boxplus + \frac{1}{\sqrt{2}} \cdot \boxminus \equiv \forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{X}$, we conclude that

$$\vdash (H)\ |+\rangle : \forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to \mathcal{Y} \to \mathcal{X}.$$

Notice that $(H)\ |+\rangle \to^* \mathbf{true}$.

2.3 Properties

As previously mentioned, λ^{vec} does not directly satisfy the standard formulation of the Subject Reduction property, but rather a weakened version of it.

Since the terms of λ^{vec} are not explicitly typed, the system is bound to have sequents such as $\Gamma \vdash \mathbf{t} : T_1$ and $\Gamma \vdash \mathbf{t} : T_2$ with distinct types T_1 and T_2 for the same term $\mathbf{t}$. Using Rules $(+_I)$ and (α_I) it is possible to obtain the valid typing judgement $\Gamma \vdash \alpha \cdot \mathbf{t} + \beta \cdot \mathbf{t} : \alpha \cdot T_1 + \beta \cdot T_2$. Given that $\alpha \cdot \mathbf{t} + \beta \cdot \mathbf{t}$ is reduced to $(\alpha + \beta) \cdot \mathbf{t}$, a regular subject reduction would ask for the valid sequent $\Gamma \vdash (\alpha + \beta) \cdot \mathbf{t} : \alpha \cdot T_1 + \beta \cdot T_2$. But since, in general, the equivalence $\alpha \cdot T_1 + \beta \cdot T_2 \equiv (\alpha + \beta) \cdot T_1 \equiv (\alpha + \beta) \cdot T_2$ is not satisfied, a workaround was needed.

To solve this problem, λ^{vec} introduced a notion of order on types. Said order, denoted with $\sqsupseteq$, was chosen so that the factorisation rules make the types of terms smaller. In particular, it satisfied that $(\alpha + \beta) \cdot T_1 \sqsupseteq \alpha \cdot T_1 + \beta \cdot T_2$ and $(\alpha + \beta) \cdot T_2 \sqsupseteq \alpha \cdot T_1 + \beta \cdot T_2$ whenever T_1 and T_2 are types for the same term. This approach was also extended to solve a second pitfall coming from the rule $\mathbf{t} + \mathbf{0} \to \mathbf{t}$. Indeed, although $x : \mathcal{X} \vdash x + \mathbf{0} : \mathcal{X} + 0 \cdot T$ is well-typed for any inhabited T, the sequent $x : \mathcal{X} \vdash x : \mathcal{X} + 0 \cdot T$ is not valid in general. Therefore, the ordering was extended to also have $\mathcal{X} \sqsupseteq \mathcal{X} + 0 \cdot T$.

Notice that this ordering did not introduce a subtyping relation. For example, although $\vdash (\alpha + \beta) \cdot \lambda x.\lambda y.x : (\alpha + \beta) \cdot \forall \mathcal{X}.\mathcal{X} \to (\mathcal{X} \to \mathcal{X})$ is valid and $(\alpha + \beta) \cdot \forall \mathcal{X}.\mathcal{X} \to (\mathcal{X} \to \mathcal{X}) \sqsupseteq \alpha \cdot \forall \mathcal{X}.\mathcal{X} \to (\mathcal{X} \to \mathcal{X}) + \beta \cdot \forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to (\mathcal{Y} \to \mathcal{Y})$, the sequent $\vdash (\alpha + \beta) \cdot \lambda x.\lambda y.x : \alpha \cdot \forall \mathcal{X}.\mathcal{X} \to (\mathcal{X} \to \mathcal{X}) + \beta \cdot \forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to (\mathcal{Y} \to \mathcal{Y})$ is not valid.

First, the (antisymmetric) ordering relation $\sqsupseteq$ is defined on types discussed above as the smallest reflexive, transitive and congruent relation satisfying the rules:

1. $(\alpha + \beta) \cdot T \sqsupseteq \alpha \cdot T + \beta \cdot T'$ if there are $\Gamma, \mathbf{t}$ such that $\Gamma \vdash \alpha \cdot \mathbf{t} : \alpha \cdot T$ and $\Gamma \vdash \beta \cdot \mathbf{t} : \beta \cdot T'$.
2. $T \sqsupseteq T + 0.R$ for any type R.
3. If $T \sqsupseteq R$ and $U \sqsupseteq V$, then $T + S \sqsupseteq R + S$, $\alpha \cdot T \sqsupseteq \alpha \cdot R$, $U \to T \sqsupseteq U \to R$ and $\forall X.U \sqsupseteq \forall X.V$.

Note the fact that $\Gamma \vdash \mathbf{t} : T$ and $\Gamma \vdash \mathbf{t} : T'$ does not imply that $\beta \cdot T \sqsupseteq \beta \cdot T'$. For instance, although $\beta \cdot T \sqsupseteq 0 \cdot T + \beta \cdot T'$, this does not mean that $0 \cdot T + \beta \cdot T' \equiv \beta \cdot T'$.

Let R be any reduction rule from Figure 2.1, and $\to_R$ a one-step reduction by rule R. A weak version of the Subject Reduction theorem can be stated as follows.

Theorem 2.3.1 (Weak Subject Reduction [3, Theorem 4.1]). *For any terms $\mathbf{t}$, $\mathbf{t}'$, any context Γ and any type T, if $\mathbf{t} \to_R \mathbf{t}'$ and $\Gamma \vdash \mathbf{t} : T$, then:*

1. *if $R \notin$ Group F, then $\Gamma \vdash \mathbf{t}' : T$;*
2. *if $R \in$ Group F, then $\exists S \sqsupseteq T$ such that $\Gamma \vdash \mathbf{t}' : S$ and $\Gamma \vdash \mathbf{t} : S$.*

□

The type system of λ^{vec} enforces Strong Normalisation of well-typed terms.

Theorem 2.3.2 (Strong Normalisation [3, Theorem 5.7]). *If $\Gamma \vdash \mathbf{t} : T$ is a valid sequent, then $\mathbf{t}$ is strongly normalising.* □

Chapter 3

Our revision

Chapter Summary

We present the revisions we made to the terms and types of λ^{vec}, and the decisions behind the new design. ■

In this chapter we present a revision of λ^{vec}, denoted as $\lambda^{\text{vec}*}$. As previously discussed, λ^{vec} is not able to satisfy the standard formulation of the Subject Reduction property. This stems from the polymorphic nature of the system, which allows a single term to have more than one type; and the fact that the reduction rules from group F collapse a sum of terms into a single one.
Consider a term $\mathbf{t}$, which types are T and R; and the term $\alpha \cdot \mathbf{t} + \beta \cdot \mathbf{t}$. The latter term would then be reduced to $(\alpha + \beta) \cdot \mathbf{t}$, but the system is unable to express all of its possible types: it will collapse the type to either $(\alpha + \beta) \cdot T$ or $(\alpha + \beta) \cdot R$, but it cannot type it with $\alpha \cdot T + \beta \cdot R$ as expected.

The revision presented in this chapter allows the system to type such terms correctly, and lays the foundation to prove that this new system satisfies the standard Subject Reduction property.

Chapter plan. In Section 2.1 we examine the terms and the reduction rules. In Section 2.2 we present the type system along with the typing rules.

3.1 The terms

We begin by presenting the untyped version of $\lambda^{\text{vec}*}$, described in Figure 3.1. In essence, the system remains the same as the original λ^{vec}, except for one major change: the removal of the term $\mathbf{0}$.

The term $\mathbf{0}$ was removed in order to simplify the system. Indeed, to guarantee the Subject Reduction property is satisfied, rules like $\mathbf{t} + \mathbf{0} \to \mathbf{t}$ required that more typing rules were added in order to relate those terms and their types.

$$
\begin{array}{rrl}
\textit{Terms:} & \mathbf{r},\mathbf{s},\mathbf{t},\mathbf{u} ::= & \mathbf{b} \mid (\mathbf{t})\ \mathbf{r} \mid \alpha\cdot\mathbf{t} \mid \mathbf{t}+\mathbf{r} \\
\textit{Basis terms:} & \mathbf{b} ::= & x \mid \lambda x.\mathbf{t}
\end{array}
$$

Group E:	*Group F:*	*Group A:*
$1\cdot\mathbf{t}\to\mathbf{t}$	$\alpha\cdot\mathbf{t}+\beta\cdot\mathbf{t}\to(\alpha+\beta)\cdot\mathbf{t}$	$(\mathbf{t}+\mathbf{r})\ \mathbf{u}\to(\mathbf{t})\ \mathbf{u}+(\mathbf{r})\ \mathbf{u}$
$\alpha\cdot(\beta\cdot\mathbf{t})\to(\alpha\times\beta)\cdot\mathbf{t}$	$\alpha\cdot\mathbf{t}+\mathbf{t}\to(\alpha+1)\cdot\mathbf{t}$	$(\mathbf{t})\ (\mathbf{r}+\mathbf{u})\to(\mathbf{t})\ \mathbf{r}+(\mathbf{t})\ \mathbf{u}$
$\alpha\cdot(\mathbf{t}+\mathbf{r})\to\alpha\cdot\mathbf{t}+\alpha\cdot\mathbf{r}$	$\mathbf{t}+\mathbf{t}\to(1+1)\cdot\mathbf{t}$	$(\alpha\cdot\mathbf{t})\ \mathbf{r}\to\alpha\cdot(\mathbf{t})\ \mathbf{r}$
	Group B:	$(\mathbf{t})\ (\alpha\cdot\mathbf{r})\to\alpha\cdot(\mathbf{t})\ \mathbf{r}$
	$(\lambda x.\mathbf{t})\ \mathbf{b}\to\mathbf{t}[\mathbf{b}/x]$	

$$
\frac{\mathbf{t}\to\mathbf{r}}{\alpha\cdot\mathbf{t}\to\alpha\cdot\mathbf{r}}\quad
\frac{\mathbf{t}\to\mathbf{r}}{\mathbf{u}+\mathbf{t}\to\mathbf{u}+\mathbf{r}}\quad
\frac{\mathbf{t}\to\mathbf{r}}{(\mathbf{u})\ \mathbf{t}\to(\mathbf{u})\ \mathbf{r}}\quad
\frac{\mathbf{t}\to\mathbf{r}}{(\mathbf{t})\ \mathbf{u}\to(\mathbf{r})\ \mathbf{u}}\quad
\frac{\mathbf{t}\to\mathbf{r}}{\lambda x.\mathbf{t}\to\lambda x.\mathbf{r}}
$$

Figure 3.1: Syntax, reduction rules and context rules of $\lambda^{\mathrm{vec}*}$.

3.2 Type system

We present the type system of $\lambda^{\mathrm{vec}*}$, described in Figure 3.2. Note that the type grammar remains the same as the original system, and only the typing rules have changed.

3.2.1 Typing rules

Since the main focus of this work is to provide a revision of λ^{vec} to recover the Subject Reduction property, we deemed necessary to revise the typing rules. We start by analysing the problem the original system had.

Consider a term $\mathbf{t}$ which types are T and R, and the term $\alpha\cdot\mathbf{t}+\beta\cdot\mathbf{t}$ of type $\alpha\cdot T+\alpha\cdot R$. Upon reducing the latter using the $\alpha\cdot\mathbf{t}+\beta\cdot\mathbf{t}\to(\alpha+\beta)\cdot\mathbf{t}$ rewrite rule, which in the original system could not be coherently typed, since the type either collapsed to $(\alpha+\beta)\cdot T$ or to $(\alpha+\beta)\cdot R$, instead of the expected $\alpha\cdot T+\beta\cdot R$.

We can generalise the problem, so for any term $\mathbf{t}$ that can be typed with $T_1,\ldots,T_n$, then the system should be able to type $(\sum_{i=1}^{n}\alpha_i)\cdot\mathbf{t}$ with $\sum_{i=1}^{n}\alpha_i\cdot T_i$. Notice that the only condition we must satisfy is that the scalar associated with the term is equal to the sum of the scalars of the type, which in this case is $\sum_{i=1}^{n}\alpha_i$.

The S rule was then introduced to solve this problem, and it also served as a replacement for the α_I rule, which can be considered a particular case of the former one, as seen below:

$$
\frac{\Gamma\vdash\mathbf{t}:T_i\ \forall i\in\{1\}}{\Gamma\vdash\alpha\cdot\mathbf{t}:\alpha\cdot T_1}\,S\equiv\frac{\Gamma\vdash\mathbf{t}:T_1}{\Gamma\vdash\alpha\cdot\mathbf{t}:\alpha\cdot T_1}\,\alpha_I
$$

However, the S rule alone is not enough to solve the problem. Continuing with the example, by applying the new S rule

$$
\frac{\dfrac{\vdots}{\Gamma\vdash\mathbf{t}:T}\quad\dfrac{\vdots}{\Gamma\vdash\mathbf{t}:R}}{\Gamma\vdash(\alpha+\beta)\cdot\mathbf{t}:\alpha\cdot T+\beta\cdot R}\,S
$$

Consider now that $\alpha+\beta=1$, so $\Gamma\vdash 1\cdot\mathbf{t}:\alpha\cdot T+\beta\cdot R$. By applying the $1\cdot\mathbf{t}\to\mathbf{t}$ rewrite rule, then we would want to derive $\Gamma\vdash\mathbf{t}:\alpha\cdot T+\beta\cdot R$, which cannot be done with the current system. The 1_E rule was introduced to solve this issue.

$$\begin{array}{llcl}
\textit{Types:} & T, R, S & ::= & U \mid \alpha \cdot T \mid T + R \mid \mathbb{X} \\
\textit{Unit types:} & U, V, W & ::= & \mathcal{X} \mid U \to T \mid \forall \mathcal{X}.U \mid \forall \mathbb{X}.U
\end{array}$$

$$\begin{array}{rclcrcl}
1 \cdot T & \equiv & T & \quad & \alpha \cdot T + \beta \cdot T & \equiv & (\alpha + \beta) \cdot T \\
\alpha \cdot (\beta \cdot T) & \equiv & (\alpha \times \beta) \cdot T & & T + R & \equiv & R + T \\
\alpha \cdot T + \alpha \cdot R & \equiv & \alpha \cdot (T + R) & & T + (R + S) & \equiv & (T + R) + S
\end{array}$$

Typing rules

$$\frac{}{\Gamma, x : U \vdash x : U}\, ax \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T}\, \to_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i) \quad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j/\vec{X}]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}]}\, \to_E$$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i \quad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n}\, \forall_I \qquad \frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X]}\, \forall_E$$

$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R}\, +_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \quad R \equiv T}{\Gamma \vdash \mathbf{t} : R}\, \equiv \qquad \frac{\Gamma \vdash 1 \cdot \mathbf{t} : T}{\Gamma \vdash \mathbf{t} : T}\, 1_E$$

$$\frac{\Gamma \vdash \mathbf{t} : T_i \quad \forall i \in \{1, \dots, n\}}{\Gamma \vdash \left(\sum_{i=1}^{n} \alpha_i\right) \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot T_i}\, S$$

Figure 3.2: Types and typing rules of $\lambda^{\text{vec}*}$. As in λ^{vec}, we use X when we do not want to specify if it is $\mathcal{X}$ or $\mathbb{X}$, that is, unit variables or general variables respectively. In $T[A/X]$, if $X = \mathcal{X}$, then A is a unit type, and if $X = \mathbb{X}$, then A can be any type. We also may write $\forall_I$ and $\forall_{\mathbb{I}}$ (resp. $\forall_{\mathcal{E}}$ and $\forall_{\mathbb{E}}$) when we need to specify which kind of variable is being used.

We also revised the $\forall$ rules, and we considered necessary to make them introduce or remove the $\forall$ to one summand at a time, instead of affecting the whole sum; thus making both rules ($\forall_I$ and $\forall_E$) less rigid. We made this change to ease some of the proofs and make them less complex. Note that since the sum of types is defined as a congruence, it is possible to modify any of the U_i by simply "swapping" the desired i^{th} unit for the n^{th} one in the sum, and then applying the $\forall$ rule.

Lastly, we removed the 0_I rule, since we do not have the term $\mathbf{0}$ anymore.

Chapter 4

Subject Reduction

Chapter Summary

We prove that the classical formulation of Subject Reduction is satisfied by λ^{vec*}*, and present several intermediate results that further characterise the system, such as the generation lemmas for every term.* ■

As previously discussed, recovering the Subject Reduction property constitutes the main focus of this work. In the original system, the Group F was the group of rules that required special consideration and did not satisfy the property in full.

We will also introduce other intermediate results that, despite being needed for the demonstration, have intrinsic value as they help further characterise the system. Such is the case of the generation lemmas which, given a sequent $\Gamma \vdash \mathbf{t} : T$, provide a characterisation of the type T based on the form of $\mathbf{t}$.

Chapter plan. In Section 4.1 we present all the intermediate results used in the Subject Reduction demonstration. In Section 4.2 we prove that the Subject Reduction theorem is satisfied by $\lambda^{\mathrm{vec}*}$.

4.1 Prerequisites for the proof

The proof of the Subject Reduction theorem requires some intermediate results that we develop in this section.
We will use the following notations:

- We use the standard notation for equivalence classes: $[x]$ identifies the equivalence class that contains the element x.
- Given a derivation tree π as following

  $$\pi = \begin{cases} \dfrac{\vdots}{\Gamma \vdash \mathbf{t} : T} \end{cases}$$

  we note that tree as $\pi = \Gamma \vdash \mathbf{t} : T$. We will also note $size(\pi)$ to the number of sequents present on it.

We show how types are characterised, in the following lemma.

Lemma 4.1.1 (Characterisation of types [3, Lemma 4.2])**.** *For any type T, there exist $n, m \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_m \in \mathsf{S}$, distinct unit types $U_1, \ldots, U_n$ and distinct general variables $\mathbb{X}_1, \ldots, \mathbb{X}_m$ such that*

$$T \equiv \sum_{i=1}^{n} \alpha_i \cdot U_i + \sum_{j=1}^{m} \beta_j \cdot \mathbb{X}_j$$

□

Our system admits weakening, as stated by the following lemma.

Lemma 4.1.2 (Weakening)**.** *Let $\mathbf{t}$ be such that $x \notin FV(\mathbf{t})$. Then $\Gamma \vdash \mathbf{t} : T$ is derivable if and only if $\Gamma, x : U \vdash \mathbf{t} : T$ is derivable.*

Proof. By a straightforward induction on the type derivation. □

The following two lemmas present some properties of the equivalence relation.

Lemma 4.1.3 (Equivalence between sums of distinct elements (up to $\equiv$) [3, Lemma 4.4])**.** *Let $U_1, \ldots, U_n$ be a set of distinct (not equivalent) unit types, and let $V_1, \ldots, V_m$ be also a set distinct unit types. If $\sum_{i=1}^{n} \alpha_i \cdot U_i \equiv \sum_{j=1}^{m} \beta_j \cdot V_j$, then $m = n$ and there exists a permutation p of m such that $\forall i$, $\alpha_i = \beta_{p(i)}$ and $U_i \equiv V_{p(i)}$.*

Lemma 4.1.4 (Equivalences $\forall$)**.** *The following equivalences hold:*

1. *$\sum_{i=1}^{n} \alpha_i \cdot U_i \equiv \sum_{j=1}^{m} \beta_j \cdot V_j$ implies there exists $k \in \{1, \ldots, m\}$ such that $\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n \equiv \sum_{j=1}^{k-1} \beta_j \cdot V_j + \beta_k \cdot \forall X.V_k + \sum_{j=k+1}^{m} \beta_j \cdot V_j$.*

2. *$\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n \equiv \sum_{j=1}^{k-1} \beta_j \cdot V_j + \beta_k \cdot \forall X.V_k + \sum_{j=k+1}^{m} \beta_j \cdot V_j$ for some $k \in \{1, \ldots, m\}$ implies that $\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X] \equiv \sum_{j=1}^{k-1} \beta_j \cdot V_j + \beta_k \cdot V_k[A/X] + \sum_{j=k+1}^{m} \beta_j \cdot V_j$.*

Proof. Item (1) ....

From Lemma 4.1.3, $m = n$ and there exists a permutation p of m such that for all $i \in \{1, \ldots, n\}$ we have that $\alpha_i = \beta_{p(i)}$ and $U_i \equiv V_{p(i)}$.
Then, $\sum_{i=1}^{n} \alpha_i \cdot U_i \equiv \sum_{i=1}^{n} \beta_{p(i)} \cdot V_{p(i)}$.
Take $k = p(n)$, therefore

$$\begin{aligned}
\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n &\equiv \sum_{i=1}^{n-1} \beta_{p(i)} \cdot V_{p(i)} + \beta_{p(n)} \cdot \forall X.V_{p(n)} \\
&\equiv \sum_{i=1}^{n-1} \beta_{p(i)} \cdot V_{p(i)} + \beta_k \cdot \forall X.V_k \\
&\equiv \sum_{j=1}^{k-1} \beta_j \cdot V_j + \beta_k \cdot \forall X.V_k + \sum_{j=k+1}^{m} \beta_j \cdot V_j
\end{aligned}$$

Item (2) ....

From Lemma 4.1.3, $m = n$ and there exists a permutation p of m such that for all $i \in \{1, \ldots, n\}$ we have that $\alpha_i = \beta_{p(i)}$, $U_i \equiv V_{p(i)}$ if $i \neq n$, and $\forall X.U_n \equiv \forall X.V_{p(n)}$.
Then, $\sum_{i=1}^{n} \alpha_i \cdot U_i = \sum_{i=1}^{n} \beta_{p(i)} \cdot V_{p(i)}$.
Without loss of generality, take $k = p(n)$, therefore

$$\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X] \equiv \sum_{i=1}^{n-1} \beta_{p(i)} \cdot V_{p(i)} + \beta_{p(n)} \cdot V_{p(n)}[A/X]$$

$$\equiv \sum_{i=1}^{n-1} \beta_{p(i)} \cdot V_{p(i)} + \beta_k \cdot V_k[A/X]$$
$$\equiv \sum_{j=1}^{k-1} \beta_j \cdot V_j + \beta_k \cdot V_k[A/X] + \sum_{j=k+1}^{m} \beta_j \cdot V_j$$

□

The next lemma extends the way the ∀ typing rules can be applied

Lemma 4.1.5. *The following statements hold:*

1. *If* $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i$ *and* $X \notin FV(\Gamma)$, *then* $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{k-1} \alpha_i \cdot U_i + \alpha_k \cdot \forall X.U_k + \sum_{i=k+1}^{n} \alpha_i \cdot U_i$, *for any* $k \in \{1, \ldots\ n\}$.
2. *If* $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{k} \alpha_i \cdot U_i + \alpha_k \cdot \forall X.U_k + \sum_{i=k+1}^{n} \alpha_i \cdot U_i$, *then* $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{k-1} \alpha_i \cdot U_i + \alpha_k \cdot U_k[A/X] + \sum_{i=k+1}^{n} \alpha_i \cdot U_i$, *for any* $k \in \{1, \ldots\ n\}$.

Proof. Consider $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i$, and the following definitions:

$$\mathcal{F}(V) = \sum_{i=1}^{k-1} \alpha_i \cdot U_i + \alpha_k \cdot V + \sum_{i=k+1}^{n} \alpha_i \cdot U_i \qquad \mathcal{G}(V) = \sum_{i=1}^{k-1} \alpha_i \cdot U_i + \sum_{i=k+1}^{n} \alpha_i \cdot U_i + \alpha_k \cdot V$$

Item (1) ....

Taking any $k \in \{1, \ldots\ n\}$, we have

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i \quad X \notin FV(\Gamma) \quad \sum_{i=1}^{n} \alpha_i \cdot U_i \equiv \mathcal{G}(U_k)}{\Gamma \vdash \mathbf{t} : \mathcal{G}(U_k)} \equiv}{\Gamma \vdash \mathbf{t} : \mathcal{G}(\forall X.U_k)} \forall_I \quad \mathcal{G}(\forall X.U_k) \equiv \mathcal{F}(\forall X.U_k)}{\Gamma \vdash \mathbf{t} : \mathcal{F}(\forall X.U_k) = \sum_{i=1}^{k-1} \alpha_i \cdot U_i + \alpha_k \cdot \forall X.U_k + \sum_{i=k+1}^{n} \alpha_i \cdot U_i} \equiv$$

Item (2) ....

Taking any $k \in \{1, \ldots\ n\}$, we have

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i = \mathcal{F}(\forall X.U_k) \quad \mathcal{F}(\forall X.U_k) \equiv \mathcal{G}(\forall X.U_k)}{\Gamma \vdash \mathbf{t} : \mathcal{G}(\forall X.U_k)} \equiv}{\Gamma \vdash \mathbf{t} : \mathcal{G}(U_k[A/X])} \forall_E \quad \mathcal{G}(U_k[A/X]) \equiv \mathcal{F}(U_k[A/X])}{\Gamma \vdash \mathbf{t} : \mathcal{F}(U_k[A/X]) = \sum_{i=1}^{k-1} \alpha_i \cdot U_i + \alpha_k \cdot U_k[A/X] + \sum_{i=k+1}^{n} \alpha_i \cdot U_i} \equiv$$

□

Next, we present the relation $\succeq$ from λ^{vec} [3, Definition 4.6], which defines the relation between types of the form $\forall X.T$ and T, with some alterations.

Definition 4.1.6. *For any types T, R, and any context Γ such that for some term $\mathbf{t}$*

$$\cfrac{\cfrac{\Gamma \vdash \mathbf{t} : R}{\vdots}}{\Gamma \vdash \mathbf{t} : T}$$

1. *If $X \notin FV(\Gamma)$, write $R \prec_{X,\Gamma} T$ if either:*
   - $R \equiv \sum_{i=1}^{n} \alpha_i \cdot U_i$ *and* $T \equiv \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n$, *or*
   - $R \equiv \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n$ *and* $T \equiv \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X]$.
2. *If $\mathcal{V}$ is a set of type variables such that $\mathcal{V} \cap FV(\Gamma) = \emptyset$, we define $\preceq_{\mathcal{V},\Gamma}$ inductively:*
   - *If $R \prec_{X,\Gamma} T$, then $R \preceq_{\mathcal{V}\cup\{X\},\Gamma} T$.*
   - *If $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$, $S \preceq_{\mathcal{V}_1,\Gamma} R$ and $R \preceq_{\mathcal{V}_2,\Gamma} T$, then $S \preceq_{\mathcal{V}_1\cup\mathcal{V}_2,\Gamma} T$.*
   - *If $R \equiv T$, then $R \preceq_{\mathcal{V},\Gamma} T$.*

   Note that these relations only predicate on the types and the context, thus they hold for any term $\mathbf{t}$.

Example 4.1.7. *Consider the following derivation.*

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \vdash \mathbf{t} : T \quad T \equiv \sum_{i=1}^{n} \alpha_i \cdot U_i}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i}\equiv \qquad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n}\forall_I}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[V/X]}\forall_{\mathcal{E}} \qquad \mathbb{Y} \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall \mathbb{Y}.U_n[V/X]}\forall_{\mathbb{I}} \qquad \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall \mathbb{Y}.U_n[V/X] \equiv R}{\Gamma \vdash \mathbf{t} : R}\equiv$$

Then $R \preceq_{\{X,\mathbb{Y}\},\Gamma} T$.

Lemma 4.1.8. *For any unit type $U \not\equiv \forall X.V$, if $U \preceq_{\mathcal{V},\Gamma} \forall X.V$, then $X \notin FV(\Gamma)$.*

Proof. By definition of $\preceq$. □

The following lemma states that if two arrow types are ordered, then they are equivalent up to some substitution.

Lemma 4.1.9 (Arrows comparison)**.** *$V \to R \preceq_{\mathcal{V},\Gamma} \forall \vec{X}.(U \to T)$, then $U \to T \equiv (V \to R)[\vec{A}/\vec{Y}]$, with $\vec{Y} \notin FV(\Gamma)$.*

Proof. Let $(\ \cdot\ )^\circ$ be a map from types to types defined as follows,

$$X^\circ = X \qquad (U \to T)^\circ = U \to T \qquad (\forall X.T)^\circ = T^\circ$$
$$(\alpha \cdot T)^\circ = \alpha \cdot T^\circ \qquad (T + R)^\circ = T^\circ + R^\circ$$

We need three intermediate results:

1. If $T \equiv R$, then $T^\circ \equiv R^\circ$.
2. For any types U, A, there exists B such that $(U[A/X])^\circ = U^\circ[B/X]$.
3. For any types V, U, there exists $\vec{A}$ such that if $V \preceq_{\mathcal{V},\Gamma} \forall \vec{X}.U$, then $U^\circ \equiv V^\circ[\vec{A}/\vec{X}]$.

Proofs.

1. Induction on the equivalence rules. We only give the basic cases since the inductive step, given by the context where the equivalence is applied, is trivial.
   - $(1 \cdot T)^\circ = 1 \cdot T^\circ \equiv T^\circ$.
   - $(\alpha \cdot (\beta \cdot T))^\circ = \alpha \cdot (\beta \cdot T^\circ) \equiv (\alpha \times \beta) \cdot T^\circ = ((\alpha \times \beta) \cdot T)^\circ$.
   - $(\alpha \cdot T + \alpha \cdot R)^\circ = \alpha \cdot T^\circ + \alpha \cdot R^\circ \equiv \alpha \cdot (T^\circ + R^\circ) = (\alpha \cdot (T + R))^\circ$.
   - $(\alpha \cdot T + \beta \cdot T)^\circ = \alpha \cdot T^\circ + \beta \cdot T^\circ \equiv (\alpha + \beta) \cdot T^\circ = ((\alpha + \beta) \cdot T)^\circ$.
   - $(T + R)^\circ = T^\circ + R^\circ \equiv R^\circ + T^\circ = (R + T)^\circ$.
   - $(T + (R + S))^\circ = T^\circ + (R^\circ + S^\circ) \equiv (T^\circ + R^\circ) + S^\circ = ((T + R) + S)^\circ$.
2. Structural induction on U.
   - $U = \mathcal{X}$. Then $(\mathcal{X}[V/X])^\circ = V^\circ = \mathcal{X}[V^\circ/X] = \mathcal{X}^\circ[V^\circ/X]$.
   - $U = \mathcal{Y}$. Then $(\mathcal{Y}[A/X])^\circ = \mathcal{Y} = \mathcal{Y}^\circ[A/X]$.
   - $U = V \to T$. Then $((V \to T)[A/X])^\circ = (V[A/X] \to T[A/X])^\circ = V[A/X] \to T[A/X] = (V \to T)[A/X] = (V \to T)^\circ[A/X]$.
   - $U = \forall Y.V$. Then $((\forall Y.V)[A/X])^\circ = (\forall Y.V[A/X])^\circ = (V[A/X])^\circ$, which by the induction hypothesis is equivalent to $V^\circ[B/X] = (\forall Y.V)^\circ[B/X]$.
3. It suffices to show this for $V \prec_{X,\Gamma} \forall \vec{X}.U$. Cases:
   - $\forall \vec{X}.U \equiv \forall Y.V$, then notice that $(\forall \vec{X}.U)^\circ \equiv_{(1)} (\forall Y.V)^\circ = V^\circ$.
   - $V \equiv \forall Y.W$ and $\forall \vec{X}.U \equiv W[A/X]$, then $(\forall \vec{X}.U)^\circ \equiv_{(1)} (W[A/X])^\circ \equiv_{(2)} W^\circ[B/X] = (\forall Y.W)^\circ[B/X] \equiv_{(1)} V^\circ[B/X]$.

Proof of the lemma. $U \to T \equiv (U \to T)^\circ$, by the intermediate result 3, this is equivalent to $(V \to R)^\circ[\vec{A}/\vec{X}] = (V \to R)[\vec{A}/\vec{X}]$. □

The following lemmas express the formal relation between the terms and their types.

Five generation lemmas are required: two classical ones, for applications (Lemma 4.1.12) and abstractions (Lemma 4.1.13); and three new ones for scalars (Lemma 4.1.10), sums (Lemma 4.1.11) and basis terms (Lemma 4.1.14).

Lemma 4.1.10 (Scalars)**.** *For any context Γ, term $\mathbf{t}$, type T, if $\pi = \Gamma \vdash \alpha \cdot \mathbf{t} : T$, there exist $R_1, \ldots, R_n$, $\alpha_1, \ldots, \alpha_n$ such that*

- $T \equiv \sum_{i=1}^n \alpha_i \cdot R_i$.
- $\pi_i = \Gamma \vdash \mathbf{t} : R_i$, *with* $size(\pi) > size(\pi_i)$, *for* $i \in \{1, \ldots, n\}$.
- $\sum_{i=1}^n \alpha_i = \alpha$.

Proof. By induction on the typing derivation

Case S

$$\frac{\Gamma \vdash \mathbf{t} : T_i \ \forall i \in \{1, \dots, n\}}{\Gamma \vdash \left(\sum_{i=1}^{n} \alpha_i\right) \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot T_i} S$$

Trivial case.

Case $\equiv$

$$\frac{\pi' = \Gamma \vdash \alpha \cdot \mathbf{t} : T \quad T \equiv R}{\pi = \Gamma \vdash \alpha \cdot \mathbf{t} : R} \equiv$$

By the induction hypothesis there exist $S_1, \dots, S_n$, $\alpha_1, \dots, \alpha_n$ such that

- $T \equiv R \equiv \sum_{i=1}^{n} \alpha_i \cdot S_i$.
- $\pi_i = \Gamma \vdash \mathbf{t} : S_i$, with $size(\pi') > size(\pi_i)$, for $i \in \{1, \dots, n\}$.
- $\sum_{i=1}^{n} \alpha_i = \alpha$.

It is easy to see that $size(\pi) > size(\pi')$, so the lemma holds.

Case 1_E

$$\frac{\pi = \Gamma \vdash 1 \cdot (\alpha \cdot \mathbf{t}) : T}{\Gamma \vdash \alpha \cdot \mathbf{t} : T} 1_E$$

By induction hypothesis, there exist $R_1, \dots, R_m$, $\beta_1, \dots, \beta_m$ such that

- $T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j$.
- $\pi_j = \Gamma \vdash \alpha \cdot \mathbf{t} : R_j$ with $size(\pi) > size(\pi_j)$ for $j = \{1, \dots, m\}$.
- $\sum_{j=1}^{m} \beta_j = 1$.

Since $size(\pi) > size(\pi_j)$, then by applying the induction hypothesis again for all $j = \{1, \dots, m\}$, we have that there exist $S_{(j,1)}, \dots, S_{(j,n_j)}$, $\alpha_{(j,1)}, \dots, \alpha_{(j,n_j)}$ such that

- $R_j \equiv \sum_{i=1}^{n_j} \alpha_{(j,i)} \cdot S_{(j,i)}$.
- $\pi_{(j,i)} = \Gamma \vdash \mathbf{t} : S_{(j,i)}$ with $size(\pi_j) > size(\pi_{(j,i)})$ for $i \in \{1, \dots, n_j\}$.
- $\sum_{i=1}^{n_j} \alpha_{(j,i)} = \alpha$.

Given that $\Gamma \vdash \alpha \cdot \mathbf{t} : T$, then

$$T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j \equiv \sum_{j=1}^{m} \beta_j \cdot \sum_{i=1}^{n} \alpha_{(j,i)} \cdot S_{(j,i)} \equiv \sum_{j=1}^{m} \sum_{i=1}^{n} (\beta_j \times \alpha_{(j,i)}) \cdot S_{(j,i)}$$

Finally, we must prove that $\sum_{j=1}^{m} \sum_{i=1}^{n} (\beta_j \times \alpha_{(j,i)}) = \alpha$,

$$\sum_{j=1}^{m} \sum_{i=1}^{n} (\beta_j \times \alpha_{(j,i)}) = \sum_{j=1}^{m} \beta_j \cdot \underbrace{\sum_{i=1}^{n} \alpha_{(j,i)}}_{= \alpha} = \sum_{j=1}^{m} \beta_j \cdot \alpha = \alpha \cdot \underbrace{\sum_{j=1}^{m} \beta_j}_{= 1} = \alpha$$

Case $\forall_I$

$$\cfrac{\pi' = \Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i \quad X \notin FV(\Gamma)}{\Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n}\ \forall_I$$

By the induction hypothesis there exist $R_1, \dots, R_m$, $\mu_1, \dots, \mu_m$ such that

- $\sum_{i=1}^{n} \alpha_i \cdot U_i \equiv \sum_{j=1}^{m} \mu_j \cdot R_j$.
- $\pi_j = \Gamma \vdash \mathbf{t} : R_j$, with $size(\pi') > size(\pi_j)$, for $j \in \{1, \dots, m\}$.
- $\sum_{j=1}^{m} \mu_j = \alpha$

By applying Lemma 4.1.1 for all $j \in \{1, \dots, m\}$, and since $\sum_{i=1}^{n} \alpha_i \cdot U_i$ does not have any general variable $\mathbb{X}$, then $R_j \equiv \sum_{k=1}^{h_j} \beta_{(j,k)} \cdot V_{(j,k)}$.
Hence $\sum_{i=1}^{n} \alpha_i \cdot U_i \equiv \sum_{j=1}^{m} \mu_j \cdot \sum_{k=1}^{h_j} \beta_{(j,k)} \cdot V_{(j,k)}$.
By Lemma 4.1.4 there exists $(e, f) \in \{(j, k) \mid j \in \{1, \dots, m\}, k \in \{1, \dots, h_j\}\}$ (without loss of generality, we take $(e, f) = (m, h_m)$) such that

$$\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \forall X.U_n \equiv \sum_{j=1}^{m-1} \mu_j \cdot \underbrace{\sum_{k=1}^{h_j} \beta_{(j,k)} \cdot V_{(j,k)}}_{\equiv R_j} + \mu_m \cdot \underbrace{\left(\sum_{k=1}^{h_m - 1} \beta_{(m,k)} \cdot V_{(m,k)} + \beta_{(m,h_m)} \cdot \forall X.V_{(m,h_m)} \right)}_{\equiv R'_m}$$

We must prove that $\Gamma \vdash \mathbf{t} : R'_m$.
Since $\Gamma \vdash \mathbf{t} : \underbrace{\sum_{k=1}^{h_m} \beta_{(m,k)} \cdot V_{(m,k)}}_{\equiv R_m}$ and $X \notin FV(\Gamma)$, then by Lemma 4.1.5, $\pi'' = \Gamma \vdash \mathbf{t} : R'_m$.

Then,

$$\cfrac{\pi_j = \Gamma \vdash \mathbf{t} : R_j \ \forall j \in \{1, \dots, m-1\} \quad \pi'' = \Gamma \vdash \mathbf{t} : R'_m}{\Gamma \vdash \alpha \cdot \mathbf{t} : \underbrace{\sum_{j=1}^{m-1} \mu_j \cdot R_j + \mu_m \cdot R'_m}_{\equiv \ \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \forall X.U_n}}\ S$$

Finally, we conclude by $\equiv$ rule that $\pi = \Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n$, where $size(\pi) > size(\pi_j)$ for all $j \in \{1, \dots, m-1\}$, and $size(\pi) > size(\pi'')$.

Case $\forall_E$

$$\cfrac{\pi' = \Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n}{\Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X]}\ \forall_E$$

By the induction hypothesis there exist $R_1, \dots, R_m$, $\mu_1, \dots, \mu_m$ such that

- $\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n \equiv \sum_{j=1}^{m} \mu_j \cdot R_j$.
- $\pi_j = \Gamma \vdash \mathbf{t} : R_j$, with $size(\pi') > size(\pi_j)$, for $j \in \{1, \dots, m\}$.

- $\sum_{j=1}^{m} \mu_j = \alpha$.

By applying Lemma 4.1.1 for all $j \in \{1, \dots, m\}$, and since $\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n$ does not have any general variable $\mathbb{X}$, then $R_j \equiv \sum_{k=1}^{h_j} \beta_{(j,k)} \cdot V_{(j,k)}$.
Hence $\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n \equiv \sum_{j=1}^{m} \mu_j \cdot \sum_{k=1}^{h_j} \beta_{(j,k)} \cdot V_{(j,k)}$.
Without loss of generality, we assume that all unit types present at both sides of the equivalence are distinct, then by Lemma 4.1.3, $n = \sum_{j=1}^{m} h_j$, and by taking a partition from $\{1, \dots, \sum_{j=1}^{m} h_j\}$ (defining an equivalence class) and the trivial permutation p of n such that $p(i) = i$ (which we will omit for readability), we have

- $\alpha_i = \mu_{[i]} \times \gamma_i$, where $\gamma_i = \beta_{\left([i], \frac{i}{[i]}\right)}$.
- $U_i \equiv V_{\left([i], \frac{i}{[i]}\right)}$, if $i \neq n$.
- $\forall X.U_n \equiv V_{\left([n], \frac{n}{[n]}\right)}$, so $V_{\left([n], \frac{n}{[n]}\right)} \equiv \forall X.W'$.

Notice that $\left([n], \frac{n}{[n]}\right) = (m, h_m)$, then

$$\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n \equiv \sum_{j=1}^{m-1} \mu_j \cdot \underbrace{\sum_{k=1}^{h_j} \beta_{(j,k)} \cdot V_{(j,k)}}_{\equiv\ R_j} + \mu_m \cdot \underbrace{\left(\sum_{k=1}^{h_m - 1} \beta_{(m,k)} \cdot V_{(m,k)} + \beta_{(m,h_m)} \cdot \forall X.W' \right)}_{\equiv\ R_m}$$

By Lemma 4.1.4 there exists $(e, f) \in \{(j,k) \mid j \in \{1, \dots, m\}, k \in \{1, \dots, h_j\}\}$ (without loss of generality, we take $(e, f) = (m, h_m)$), such that

$$\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X] \equiv \sum_{j=1}^{m-1} \mu_j \cdot \underbrace{\sum_{k=1}^{h} \beta_{(j,k)} \cdot V_{(j,k)}}_{R_j} + \mu_m \cdot \underbrace{\left(\sum_{k=1}^{h_m - 1} \beta_{(m,k)} \cdot V_{(m,k)} + \beta_{(m,h_m)} \cdot W'[A/X] \right)}_{\equiv\ R'_m}$$

We must prove that $\Gamma \vdash \mathbf{t} : R'_m$.
Since $\Gamma \vdash \mathbf{t} : R_m$, then by Lemma 4.1.5, $\pi'' = \Gamma \vdash \mathbf{t} : R'_m$.
Then,

$$\frac{\pi_j = \Gamma \vdash \mathbf{t} : R_j \ \forall j \in \{1, \dots, m-1\} \quad \pi'' = \Gamma \vdash \mathbf{t} : R'_m}{\Gamma \vdash \alpha \cdot \mathbf{t} : \underbrace{\sum_{j=1}^{m-1} \mu_j \cdot R_j + \mu_m \cdot R'_m}_{\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X]}}\ S$$

Finally, we conclude by $\equiv$ rule that $\pi = \Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X]$, where $size(\pi) > size(\pi_j)$ for all $j \in \{1, \dots, m-1\}$, and $size(\pi) > size(\pi'')$. □

Lemma 4.1.11 (Sums). *If $\Gamma \vdash \mathbf{t} + \mathbf{r} : S$, there exist R, T such that*

- $S \equiv T + R$.
- $\Gamma \vdash \mathbf{t} : T$.
- $\Gamma \vdash \mathbf{r} : R$.

Proof. By induction on the typing derivation

Case $+_I$

$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R} +_I$$

Trivial.

Case $\equiv$

$$\frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : P \quad S \equiv P}{\Gamma \vdash \mathbf{t} + \mathbf{r} : S} \equiv$$

By the induction hypothesis, $S \equiv P \equiv T + R$.

Case 1_E

$$\frac{\pi = \Gamma \vdash 1 \cdot (\mathbf{t} + \mathbf{r}) : T}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T} 1_E$$

By Lemma 4.1.10, there exist $R_1, \ldots, R_m$, $\beta_1, \ldots, \beta_m$ such that

- $T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j$.
- $\pi_j = \Gamma \vdash \mathbf{t} + \mathbf{r} : R_j$ with $size(\pi) > size(\pi_j)$ for $j \in \{1, \ldots, m\}$.
- $\sum_{j=1}^{m} \beta_j = 1$

Since $size(\pi) > size(\pi_j)$, by applying the induction hypothesis for all $j \in \{1, \ldots, m\}$,

- $R_j \equiv S_{(j,1)} + S_{(j,2)}$.
- $\Gamma \vdash \mathbf{t} : S_{(j,1)}$.
- $\Gamma \vdash \mathbf{r} : S_{(j,2)}$.

Then,

$$T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j \equiv \sum_{j=1}^{m} \beta_j \cdot (S_{(j,1)} + S_{(j,2)}) \equiv \sum_{j=1}^{m} \beta_j \cdot S_{(j,1)} + \sum_{j=1}^{m} \beta_j \cdot S_{(j,2)}$$

We can rewrite T as follows:

$$P_1 = \sum_{j=1}^{m} \beta_j \cdot S_{(j,1)} \qquad P_2 = \sum_{j=1}^{m} \beta_j \cdot S_{(j,2)} \qquad T \equiv P_1 + P_2$$

Finally, we must prove that $\Gamma \vdash \mathbf{t} : P_1$ and $\Gamma \vdash \mathbf{r} : P_2$.
Since $\Gamma \vdash \mathbf{t} : S_{(j,1)}$ and $\Gamma \vdash \mathbf{r} : S_{(j,2)}$ for all $j \in \{1, \ldots, m\}$, applying the S rule in both cases we have

$$\frac{\Gamma \vdash \mathbf{t} : S_{(j,1)} \ \forall j \in \{1, \ldots, m\}}{\Gamma \vdash 1 \cdot \mathbf{t} : P_1} S \qquad \frac{\Gamma \vdash \mathbf{t} : S_{(j,2)} \ \forall j \in \{1, \ldots, m\}}{\Gamma \vdash 1 \cdot \mathbf{r} : P_2} S$$

Applying the 1_E rule to both sequents, we have

$$\Gamma \vdash \mathbf{t} : P_1 \qquad \Gamma \vdash \mathbf{r} : P_2$$

Finally, by $\equiv$ rule, $\Gamma \vdash \mathbf{t} + \mathbf{r} : T$.

Case $\forall$

$$\frac{\Gamma \vdash \mathbf{t} + \mathbf{r} : \sum_{i=1}^{n} \alpha_i \cdot U_i}{\Gamma \vdash \mathbf{t} + \mathbf{r} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot V_n} \forall$$

Rules $\forall_I$ and $\forall_E$ both have the same structure as shown above. In any case, by the induction hypothesis $\Gamma \vdash \mathbf{t} : T$ and $\Gamma \vdash \mathbf{r} : R$ with $T + R \equiv \sum_{i=1}^{n} \alpha_i \cdot U_i$.
Then, there exist $N, M \subseteq \{1, \ldots, n\}$ with $N \cup M = \{1, \ldots, n\}$ such that

$$T \equiv \sum_{i \in N \setminus M} \alpha_i \cdot U_i + \sum_{i \in N \cap M} \alpha'_i \cdot U_i \qquad \text{and} \qquad R \equiv \sum_{i \in M \setminus N} \alpha_i \cdot U_i + \sum_{i \in N \cap M} \alpha''_i \cdot U_i$$

where $\forall i \in N \cap M$, $\alpha'_i + \alpha''_i = \alpha_i$.
Therefore, using $\equiv$ (if needed) and the same $\forall$-rule, we get three possible cases:

1. $n \in N \setminus M$
   In this case, then $\Gamma \vdash \mathbf{t} : T \equiv \sum_{i \in (N \setminus M) \setminus \{n\}} \alpha_i \cdot U_i + \sum_{i \in N \cap M} \alpha'_i \cdot U_i + \alpha_n \cdot U_n$, and by applying the $\forall$-rule, we get $\Gamma \vdash \mathbf{t} : T \equiv \sum_{i \in (N \setminus M) \setminus \{n\}} \alpha_i \cdot U_i + \sum_{i \in N \cap M} \alpha'_i \cdot U_i + \alpha_n \cdot V_n$ and $\Gamma \vdash \mathbf{r} : R \equiv \sum_{i \in M \setminus N} \alpha_i \cdot U_i + \sum_{i \in N \cap M} \alpha''_i \cdot U_i$.

2. $n \in M \setminus N$
   Analogous to the previous case.

3. $n \in N \cap M$
   In this case, then $\Gamma \vdash \mathbf{t} : T \equiv \sum_{i \in N \setminus M} \alpha_i \cdot U_i + \sum_{i \in (N \cap M) \setminus \{n\}} \alpha'_i \cdot U_i + \alpha'_n \cdot U_n$ and $\Gamma \vdash \mathbf{r} : R \equiv \sum_{i \in M \setminus N} \alpha_i \cdot U_i + \sum_{i \in (N \cap M) \setminus \{n\}} \alpha''_i \cdot U_i + \alpha''_n \cdot U_n$, and by applying the $\forall$-rule, we get $\Gamma \vdash \mathbf{t} : T \equiv \sum_{i \in N \setminus M)} \alpha_i \cdot U_i + \sum_{i \in (N \cap M) \setminus \{n\}} \alpha'_i \cdot U_i + \alpha'_n \cdot V_n$ and $\Gamma \vdash \mathbf{r} : R \equiv \sum_{i \in M \setminus N)} \alpha_i \cdot U_i + \sum_{i \in (N \cap M) \setminus \{n\}} \alpha''_i \cdot U_i + \alpha''_n \cdot V_i$. ☐

Lemma 4.1.12 (Application)**.** *If $\Gamma \vdash (\mathbf{t})\ \mathbf{r} : T$, there exist $R_1, \ldots, R_h$, $\mu_1, \ldots, \mu_h$, $\mathcal{V}_1, \ldots, \mathcal{V}_h$ such that $T \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \ldots, h\}$*

- $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\Gamma \vdash \mathbf{r} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_{(k,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k, \Gamma} R_k$.

Proof. By induction on the typing derivation

Case $\to_E$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i) \quad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j/\vec{X}]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}]} \to_E$$

Take $\mu_1, \ldots, \mu_h$ such that $\sum_{k=1}^{h} \mu_k = 1$, then

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}] \equiv \sum_{k=1}^{h} \mu_k \cdot \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}]$$

So this is the trivial case.

Case $\equiv$

$$\frac{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : P \quad S \equiv P}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : S} \equiv$$

By the induction hypothesis, there exist $R_1, \ldots, R_h$, $\mu_1, \ldots, \mu_h$, $\mathcal{V}_1, \ldots, \mathcal{V}_h$ such that $P \equiv S \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \ldots, h\}$,

- $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\Gamma \vdash \mathbf{r} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_{(k,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$.

So the lemma holds.

Case 1_E

$$\frac{\pi = \Gamma \vdash 1 \cdot (\mathbf{t})\ \mathbf{r} : T}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : T} 1_E$$

By Lemma 4.1.10, there exist $R_1, \ldots, R_h$, $\mu_1, \ldots, \mu_h$ such that

- $T \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$.
- $\pi_k = \Gamma \vdash (\mathbf{t})\ \mathbf{r} : R_k$, with $size(\pi) > size(\pi_k)$, for $k \in \{1, \ldots, h\}$..
- $\sum_{k=1}^{h} \mu_k = 1$.

Since $size(\pi) > size(\pi_k)$, we apply the inductive hypothesis for all $k \in \{1, \ldots, h\}$ (and omiting the k index for readability), so there exist $S_1, \ldots, S_p$, $\eta_1, \ldots, \eta_p$, $\mathcal{V}_1, \ldots, \mathcal{V}_p$ such that $R \equiv \sum_{q=1}^{p} \eta_q \cdot S_q$, $\sum_{q=1}^{p} \eta_q = 1$ and for all $q \in \{1, \ldots, p\}$,

- $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n_q} \alpha_{(q,i)} \cdot \forall \vec{X}.(U \to T_{(q,i)})$.
- $\Gamma \vdash \mathbf{r} : \sum_{j=1}^{m_q} \beta_{(q,j)} \cdot U[\vec{A}_{(q,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_q} \sum_{j=1}^{m_q} \alpha_{(q,i)} \times \beta_{(q,j)} \cdot T_{(q,i)}[\vec{A}_{(q,j)}/\vec{X}] \preceq_{\mathcal{V}_q,\Gamma} S_q$.

Then

$$T \equiv \sum_{k=1}^{h} \mu_k \cdot R_k \equiv \sum_{k=1}^{h} \mu_k \cdot \sum_{q=1}^{p_k} \eta_{(k,q)} \cdot S_{(k,q)} \equiv \sum_{k=1}^{h} \sum_{q=1}^{p_k} (\mu_k \times \eta_{(k,q)}) \cdot S_{(k,q)}$$

Finally, we must prove that $\sum_{k=1}^{h} \sum_{q=1}^{p_k} (\mu_k \times \eta_{(k,q)}) = 1$,

$$\sum_{k=1}^{h} \sum_{q=1}^{p_k} (\mu_k \times \eta_{(k,q)}) = \sum_{k=1}^{h} \mu_k \cdot \underbrace{\sum_{q=1}^{p_k} \eta_{(k,q)}}_{=1} = \sum_{k=1}^{h} \mu_k = 1$$

Case $\forall_I$

$$\dfrac{\pi' = \Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{a=1}^{b} \sigma_a \cdot V_a \quad X \notin FV(\Gamma)}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot \forall X.V_b}\ \forall_I$$

By the induction hypothesis there exist $R_1, \ldots, R_h$, $\mu_1, \ldots, \mu_h$, $\mathcal{V}_1, \ldots, \mathcal{V}_h$ such that $\sum_{a=1}^{b} \sigma_a \cdot V_a \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \ldots, h\}$,

- $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\Gamma \vdash \mathbf{r} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_{(k,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$.

By Lemma 4.1.1, and since $\sum_{a=1}^{b} \sigma_a \cdot V_a$ does not have any general variable, then for all $k \in \{1, \ldots, h\}$, $R_k \equiv \sum_{c=1}^{d_k} \eta_{(k,c)} \cdot W_{(k,c)}$.
Hence $\sum_{a=1}^{b} \sigma_a \cdot V_a \equiv \sum_{k=1}^{h} \mu_h \cdot \sum_{c=1}^{d_k} \eta_{(k,c)} \cdot W_{(k,c)}$.
Then by Lemma 4.1.4 there exists $(e, f) \in \{(k, c) \mid k \in \{1, \ldots, h\}, c \in \{1, \ldots, d_k\}\}$ (without loss of generality, we take $(e, f) = (h, d_h)$) such that

$$\sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot \forall X.V_b \equiv \sum_{k=1}^{h-1} \mu_k \cdot \underbrace{\sum_{c=1}^{d_k} \eta_{(k,c)} \cdot W_{(k,c)}}_{R_k} + \mu_h \cdot \underbrace{\left(\sum_{c=1}^{d_h - 1} \eta_{(h,c)} \cdot W_{(h,c)} + \eta_{(h,d_h)} \cdot \forall X.W_{(h,d_h)} \right)}_{R'_h}$$

Finally, we must prove that $\sum_{i=1}^{n_h} \sum_{j=1}^{m_h} \alpha_{(h,i)} \times \beta_{(h,j)} \cdot T_{(h,i)}[\vec{A}_{(h,j)}/\vec{X}] \preceq_{\mathcal{V}'_h,\Gamma} R'_h$.
Notice that $R_h \preceq_{\mathcal{V}_h \cup \{X\},\Gamma} R'_h$, then by definition of $\preceq$, taking $\mathcal{V}'_h = \mathcal{V}_h \cup \{X\}$,
$\sum_{i=1}^{n_h} \sum_{j=1}^{m_h} \alpha_{(h,i)} \times \beta_{(h,j)} \cdot T_{(h,i)}[\vec{A}_{(h,j)}/\vec{X}] \preceq_{\mathcal{V}'_h,\Gamma} R'_h$.

Case $\forall_E$

$$\dfrac{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot \forall X.V_b}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot V_b[A/X]}\ \forall_E$$

By the induction hypothesis there exist $R_1, \ldots, R_h$, $\mu_1, \ldots, \mu_h$, $\mathcal{V}_1, \ldots, \mathcal{V}_h$ such that $\sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot \forall X.V_b \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \ldots, h\}$,

- $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\Gamma \vdash \mathbf{r} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_{(k,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$.

By Lemma 4.1.1, and since $\sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot \forall X.V_b$ does not have any general variable, $R_k \equiv \sum_{c=1}^{d_k} \eta_{(k,c)} \cdot W_{(k,c)}$.
Hence $\sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot \forall X.V_b \equiv \sum_{k=1}^{h} \mu_k \cdot \sum_{c=1}^{d_k} \eta_{(k,c)} \cdot W_{(k,c)}$.
Without loss of generality, we assume that all unit types present at both sides of the equivalence

are distinct, then by Lemma 4.1.3, $b = \sum_{k=1}^{h} d_k$, and by taking a partition from $\{1, \ldots, \sum_{k=1}^{h} d_k\}$ (defining an equivalence class) and the trivial permutation p of b such that $p(a) = a$ (which we will omit for readability), we have that

- $\sigma_a = \mu_{[a]} \times \gamma_a$, where $\gamma_a = \eta_{\left([a], \frac{a}{[a]}\right)}$.
- $V_a \equiv W_{\left([a], \frac{a}{[a]}\right)}$, if $a \neq b$.
- $\forall X.V_b \equiv W_{\left([b], \frac{b}{[b]}\right)}$, so $W_{\left([b], \frac{b}{[b]}\right)} = \forall X.W'$.

Notice that $\left([b], \frac{b}{[b]}\right) = (h, d_h)$, then

$$\sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot \forall X.V_b \equiv \sum_{k=1}^{h-1} \mu_k \cdot \underbrace{\sum_{c=1}^{d_k} \eta_{(k,c)} \cdot W_{(k,c)}}_{R_k} + \mu_h \cdot \underbrace{\left(\sum_{c=1}^{d_h - 1} \eta_{(h,c)} \cdot W_{(h,c)} + \beta_{(h,d_h)} \cdot forallX.W' \right)}_{R_h}$$

By Lemma 4.1.4 there exists $(e, f) \in \{(k, c) \mid k \in \{1, \ldots, h\}, c \in \{1, \ldots, d_k\}\}$ (without loss of generality, we take $(e, f) = (h, d_h)$) such that

$$\sum_{a=1}^{b-1} \sigma_a \cdot V_a + \sigma_b \cdot V_b[A/X] \equiv \sum_{k=1}^{h-1} \mu_j \cdot \underbrace{\sum_{c=1}^{d_k} \eta_{(k,c)} \cdot W_{(k,c)}}_{R_k} + \mu_h \cdot \underbrace{\left(\sum_{c=1}^{d_h - 1} \eta_{(h,c)} \cdot W_{(h,c)} + \beta_{(h,d_h)} \cdot W'[A/X] \right)}_{R'_h}$$

Finally, we must prove that $\sum_{i=1}^{n_h} \sum_{j=1}^{m_h} \alpha_{(h,i)} \times \beta_{(h,j)} \cdot T_{(h,i)}[\vec{A}_{(h,j)}/\vec{X}] \preceq_{\mathcal{V}'_h, \Gamma} R'_h$.
Notice that $R_h \preceq_{\mathcal{V}_h, \Gamma} R'_h$, then taking $\mathcal{V}'_h = \mathcal{V}_h$,
$\sum_{i=1}^{n_h} \sum_{j=1}^{m_h} \alpha_{(h,i)} \times \beta_{(h,j)} \cdot T_{(h,i)}[\vec{A}_{(h,j)}/\vec{X}] \preceq_{\mathcal{V}_h, \Gamma} R'_h$. □

Lemma 4.1.13 (Abstractions)**.** *If $\Gamma \vdash \lambda x.\mathbf{t} : T$, then there exist $T_1, \ldots, T_n$, $R_1, \ldots, R_n$, $U_1, \ldots, U_n$, $\alpha_1, \ldots, \alpha_n$, $\mathcal{V}_1, \ldots, \mathcal{V}_n$ such that $T \equiv \sum_{i=1}^{n} \alpha_i \cdot T_i$, $\sum_{i=1}^{n} \alpha_i = 1$ and for all $i \in \{1, \ldots, n\}$,*

- *$\Gamma, x : U_i \vdash \mathbf{t} : R_i$.*
- *$U_i \to R_i \preceq_{\mathcal{V}_i, \Gamma} T_i$.*

Proof. By induction on the typing derivation

Case $\to_I$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : R}{\Gamma \vdash \lambda x.\mathbf{t} : U \to R} \to_I$$

Trivial.

Case $\equiv$

$$\frac{\Gamma \vdash \lambda x.\mathbf{t} : R \quad R \equiv T}{\Gamma \vdash \lambda x.\mathbf{t} : T} \equiv$$

By the induction hypothesis, there exist $T_1, \ldots, T_n, R_1, \ldots, R_n, U_1, \ldots, U_n, \alpha_1, \ldots, \alpha_n, \mathcal{V}_1, \ldots, \mathcal{V}_n$ such that $T \equiv R \equiv \sum_{i=1}^{n} \alpha_i \cdot T_i$, $\sum_{i=1}^{n} \alpha_i = 1$ and for all $i \in \{1, \ldots, n\}$,

- $\Gamma, x : U_i \vdash \mathbf{t} : R_i$.
- $U_i \to R_i \preceq_{\mathcal{V}_i,\Gamma} T_i$.

So the lemma holds.

Case 1_E

$$\frac{\pi = \Gamma \vdash 1 \cdot (\lambda x.\mathbf{t}) : T}{\Gamma \vdash \lambda x.\mathbf{t} : T}\, 1_E$$

By Lemma 4.1.10, there exist $R_1, \ldots, R_m$, $\beta_1, \ldots, \beta_m$ such that

- $T \equiv \sum_{j=1}^{m} \beta_i \cdot R_j$.
- $\pi_i = \Gamma \vdash \mathbf{t} : R_j$, with $size(\pi) > size(\pi_j)$, for $j \in \{1, \ldots, n\}$.
- $\sum_{j=1}^{n} \beta_i = 1$.

Since $size(\pi) > size(\pi_j)$, by induction hypothesis, for all $j \in \{1, \ldots, n\}$ there exist $S_{(j,1)}, \ldots, S_{(j,n_j)}$, $P_{(j,1)}, \ldots, P_{(j,n_j)}$, $U_{(j,1)}, \ldots, U_{(j,n_j)}$, $\eta_{(j,1)}, \ldots, \eta_{(j,n_j)}$, $\mathcal{V}_{(j,1)}, \ldots, \mathcal{V}_{(j,n_j)}$ such that $R_j \equiv \sum_{i=1}^{n_j} \eta_{(j,i)} \cdot S_{(j,i)}$, $\sum_{i=1}^{n_j} \eta_{(j,i)} = 1$ and for all $i \in \{1, \ldots, n_j\}$,

- $\Gamma, x : U_{(j,i)} \vdash \mathbf{t} : P_{(j,i)}$.
- $U_{(j,i)} \to P_{(j,i)} \preceq_{\mathcal{V}_{(j,i)},\Gamma} S_{(j,i)}$.

Then we have

$$T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j \equiv \sum_{j=1}^{m} \beta_j \cdot \sum_{i=1}^{n_j} \eta_{(j,i)} \cdot S_{(j,i)} \equiv \sum_{j=1}^{m} \sum_{i=1}^{n_j} (\beta_j \times \eta_{(j,i)}) \cdot S_{(j,i)}$$

Finally, we must prove that $\sum_{j=1}^{m} \sum_{i=1}^{n_j} (\beta_j \times \eta_{(j,i)}) = 1$:

$$\sum_{j=1}^{m} \sum_{i=1}^{n_j} (\beta_j \times \eta_{(j,i)}) = \sum_{j=1}^{m} \beta_j \cdot \underbrace{\sum_{i=1}^{n_j} \eta_{(j,i)}}_{=\,1} = \sum_{j=1}^{m} \beta_j = 1$$

Case $\forall$

$$\frac{\Gamma \vdash \lambda x.\mathbf{t} : \sum_{k=1}^{n} \gamma_k \cdot V_k}{\Gamma \vdash \lambda x.\mathbf{t} : \sum_{k=1}^{n-1} \gamma_k \cdot V_k + \gamma_n \cdot W_n}\, \forall$$

$\forall$-rules ($\forall_I$ and $\forall_E$) both have the same structure as shown above. In both cases, by the induction hypothesis, there exist $T_1, \ldots, T_m$, $R_1, \ldots, R_m$, $U_1, \ldots, U_m$, $\alpha_1, \ldots, \alpha_m$, $\mathcal{V}_1, \ldots, \mathcal{V}_m$ such that $\sum_{k=1}^{n} \gamma_k \cdot V_k \equiv \sum_{i=1}^{m} \alpha_i \cdot T_i$, $\sum_{i=1}^{m} \alpha_i = 1$ and for all $i \in \{1, \ldots, m\}$,

- $\Gamma, x : U_i \vdash \mathbf{t} : R_i$.
- $U_i \to R_i \preceq_{\mathcal{V}_i,\Gamma} T_i$.

By Lemma 4.1.1, and since $\sum_{k=1}^{n} \gamma_k \cdot V_k$ does not have any general variable $\mathbb{X}$, then $T_i \equiv \sum_{a=1}^{b_i} \beta_{(i,a)} \cdot W'_{(i,a)}$.
Hence $\sum_{k=1}^{n} \gamma_k \cdot V_k \equiv \sum_{i=1}^{m} \alpha_i \cdot \sum_{a=1}^{b_i} \beta_{(i,a)} \cdot W'_{(i,a)}$.
By Lemma 4.1.4 there exists $(e, f) \in \{(i, a) \mid i \in \{1, \ldots, m\}, a \in \{1, \ldots, b_i\}\}$ (without loss of generality, we take $(e, f) = (m, b_m)$) such that

$$\sum_{k=1}^{n-1} \gamma_k \cdot V_k + \gamma_n \cdot W_n \equiv \sum_{i=1}^{m-1} \alpha_i \cdot \underbrace{\sum_{a=1}^{b_i} \beta_{(i,a)} \cdot W'_{(i,a)}}_{\equiv\, T_i} + \alpha_m \cdot \underbrace{\left(\sum_{a=1}^{b_m-1} \beta_{(m,a)} \cdot W'_{(m,a)} + \beta_{(m,b_m)} \cdot W''_{(m,b_m)} \right)}_{\equiv\, T'_m}$$

Finally, we must prove that $U_m \to R_m \preceq_{\mathcal{V}'_m,\Gamma} T'_m$ for some $\mathcal{V}'_m$.
Since $U_m \to R_m \preceq_{\mathcal{V}_m,\Gamma} T_m$ and $T_m \preceq_{\mathcal{V}''_m,\Gamma} T'_m$, then by $\preceq$ and using $\mathcal{V}'_m = \mathcal{V}_m \cup \mathcal{V}''_m$, we conclude that $U_m \to R_m \preceq_{\mathcal{V}'_m,\Gamma} T'_m$. □

Lemma 4.1.14 (Basis terms)**.** *For any context Γ, type T and basis term $\mathbf{b}$, if $\Gamma \vdash \mathbf{b} : T$ there exist $U_1, \ldots, U_n$, $\alpha_1, \ldots, \alpha_n$ such that*

- $T \equiv \sum_{i=1}^{n} \alpha_i \cdot U_i$.
- $\Gamma \vdash \mathbf{b} : U_i$, *for* $i \in \{1, \ldots, n\}$.
- $\sum_{i=1}^{n} \alpha_i = 1$.

Proof. By induction on the typing derivation

Case ax

$$\frac{}{\Gamma, x : U \vdash x : U}\, ax \qquad \text{and} \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T} \to_I$$

Trivial cases.

Case $\equiv$

$$\frac{\Gamma \vdash \mathbf{b} : R \quad R \equiv T}{\Gamma \vdash \mathbf{b} : T} \equiv$$

By the induction hypothesis, there exist $U_1, \ldots, U_n$, $\alpha_1, \ldots, \alpha_n$ such that

- $T \equiv R \equiv \sum_{i=1}^{n} \alpha_i \cdot U_i$.
- $\Gamma \vdash \mathbf{b} : U_i$, for $i \in \{1, \ldots, n\}$.
- $\sum_{i=1}^{n} \alpha_i = 1$.

So the lemma holds.

Case 1_E

$$\frac{\pi = \Gamma \vdash 1 \cdot \mathbf{b} : T}{\Gamma \vdash \mathbf{b} : T} 1_E$$

By Lemma 4.1.10, there exist $R_1, \ldots, R_m$, $\beta_1, \ldots, \beta_m$ such that

- $T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j$.

- $\sum_{j=1}^{m} \beta_j = 1$, and $\pi_j = \Gamma \vdash \mathbf{b} : R_j$ with $size(\pi) > size(\pi_j)$ for $j = \{1, \ldots, m\}$.
- $\sum_{j=1}^{m} \beta_j = 1$.

Since $size(\pi) > size(\pi_j)$, by induction hypothesis, for all $j = \{1, \ldots, m\}$ there exist $U_{(j,1)}, \ldots, U_{(j,n_j)}$, $\alpha_{(j,1)}, \ldots, \alpha_{(j,n_j)}$ such that

- $R_j \equiv \sum_{i=1}^{n_j} \alpha_{(j,i)} \cdot U_{(j,i)}$.
- $\Gamma \vdash \mathbf{b} : U_{(j,i)}$, for $i \in \{1, \ldots, n_j\}$.
- $\sum_{i=1}^{n_j} \alpha_{(j,i)} = 1$.

Then

$$T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j \equiv \sum_{j=1}^{m} \beta_j \cdot \sum_{i=1}^{n_j} \alpha_{(j,i)} \cdot U_{(j,i)} \equiv \sum_{j=1}^{m} \sum_{i=1}^{n_j} (\beta_j \times \alpha_{(j,i)}) \cdot U_{(j,i)}$$

Finally, we must prove that $\sum_{j=1}^{m} \sum_{i=1}^{n_j} (\beta_j \times \alpha_{(j,i)}) = 1$:

$$\sum_{j=1}^{m} \sum_{i=1}^{n_j} (\beta_j \times \alpha_{(j,i)}) = \sum_{j=1}^{m} \beta_j \cdot \underbrace{\sum_{i=1}^{n_j} \alpha_{(j,i)}}_{=\,1} = \sum_{j=1}^{m} \beta_j = 1$$

Case $\forall$

$$\frac{\Gamma \vdash \mathbf{b} : \sum_{k=1}^{n} \gamma_k \cdot V_k}{\Gamma \vdash \mathbf{b} : \sum_{k=1}^{n-1} \gamma_k \cdot V_k + \gamma_n \cdot W_n}\ \forall$$

$\forall$-rules ($\forall_I$ and $\forall_E$) both have the same structure as shown above.
In both cases, by the induction hypothesis, there exist $U_1, \ldots, U_n$, $\alpha_1, \ldots, \alpha_n$ such that

- $\sum_{k=1}^{n} \gamma_k \cdot V_k \equiv \sum_{i=1}^{n} \alpha_i \cdot U_i$.
- $\Gamma \vdash \mathbf{b} : U_i$, for $i \in \{1, \ldots, n\}$.
- $\sum_{i=1}^{n} \alpha_i = 1$.

Without loss of generality, we assume that all unit types present at both sides of the equivalence are distinct, so by Lemma 4.1.3, then $m = n$ and there exists a permutation p of m such that for all $i \in \{1, \ldots, n\}$, then $V_i = U_{p(i)}$ and $\gamma_i = \alpha_{p(i)}$, which means that $\sum_{k=1}^{n} \gamma_k = 1$.
Finally, we know that $\Gamma \vdash \mathbf{b} : V_n$, and by applying the corresponding $\forall$ rule, we have that $\Gamma \vdash \mathbf{b} : W_n$. $\square$

The following lemma ensures that by substituting type variables for types or term variables for terms in an adequate manner, the derived type is still valid.

Lemma 4.1.15 (Substitution lemma). *For any term $\mathbf{t}$, basis term $\mathbf{b}$, term variable x, context Γ, types T, U, type variable X and type A, where A is a unit type if X is a unit variable, otherwise A is a general type, we have,*

1. *if $\Gamma \vdash \mathbf{t} : T$, then $\Gamma[A/X] \vdash \mathbf{t} : T[A/X]$;*

2. if $\Gamma, x : U \vdash \mathbf{t} : T$ *and* $\Gamma \vdash \mathbf{b} : U$*, then* $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T$*.*

Proof.

Item (1) ....

Induction on the typing derivation.

.... **Case** ax ....

$$\frac{}{\Gamma, x : U \vdash x : U}\, ax$$

Notice that $\Gamma[A/X], x : U[A/X] \vdash x : U[A/X]$ can also be derived with the same rule.

.... **Case** $\to_I$ ....

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T}\to_I$$

By the induction hypothesis $\Gamma[A/X], x : U[A/X] \vdash \mathbf{t} : T[A/X]$, so by rule $\to_I$, $\Gamma[A/X] \vdash \lambda x.\mathbf{t} : U[A/X] \to T[A/X] = (U \to T)[A/X]$.

.... **Case** $\to_E$ ....

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{Y}.(U \to T_i) \quad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{m} \beta_j \cdot U[\vec{B}_j/\vec{Y}]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{B}_j/\vec{Y}]}\to_E$$

By the induction hypothesis $\Gamma[A/X] \vdash \mathbf{t} : (\sum_{i=1}^{n} \alpha_i \cdot \forall \vec{Y}.(U \to T_i))[A/X]$ and this type is equal to $\sum_{i=1}^{n} \alpha_i \cdot \forall \vec{Y}.(U[A/X] \to T_i[A/X])$. Also $\Gamma[A/X] \vdash \mathbf{r} : (\sum_{j=1}^{m} \beta_j \cdot U[\vec{B}_j/\vec{Y}])[A/X] = \sum_{j=1}^{m} \beta_j \cdot U[\vec{B}_j/\vec{Y}][A/X]$. Since $\vec{Y}$ is bound, we can consider $\vec{Y} \notin FV(A)$. Hence $U[\vec{B}_j/\vec{Y}][A/X] = U[A/X][\vec{B}_j[A/X]/\vec{Y}]$, and so, by rule $\to_E$,

$$\begin{aligned}\Gamma[A/X] \vdash (\mathbf{t})\ \mathbf{r} &: \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[A/X][\vec{B}_j[A/X]/\vec{Y}] \\ &= (\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{B}_j/\vec{Y}])[A/X]\end{aligned}$$

.... **Case** $\forall_I$ ....

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i \quad Y \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall Y.U_n}\forall_I$$

By the induction hypothesis, $\Gamma[A/X] \vdash \mathbf{t} : (\sum_{i=1}^{n} \alpha_i \cdot U_i)[A/X] = \sum_{i=1}^{n} \alpha_i \cdot U_i[A/X]$. Then, by rule $\forall_I$, $\Gamma[A/X] \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i[A/X] + \alpha_n \cdot \forall Y.U_n[A/X] = (\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall Y.U_n)[A/X]$. Since Y is bound, we can consider $Y \notin FV(A)$.

Case $\forall_E$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall Y.U_n}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[B/Y]} \forall_E$$

Since Y is bound, we can consider $Y \notin FV(A)$.
By the induction hypothesis $\Gamma[A/X] \vdash \mathbf{t} : (\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall Y.U_n)[A/X] = \sum_{i=1}^{n-1} \alpha_i \cdot U_i[A/X] + \alpha_n \cdot \forall Y.U_n[A/X]$. Then by rule $\forall_E$,
$\Gamma[A/X] \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i[A/X] + \alpha_n \cdot U_n[A/X][B/Y]$. We can consider $X \notin FV(B)$ (in other case, just take $B[A/X]$ in the $\forall$-elimination), hence

$$\begin{aligned}\sum_{i=1}^{n-1} \alpha_i \cdot U_i[A/X] + \alpha_n \cdot U_n[A/X][B/Y] &= \sum_{i=1}^{n-1} \alpha_i \cdot U_i[A/X] + \alpha_n \cdot U_n[B/Y][A/X] \\ &= (\sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[B/Y])[A/X]\end{aligned}$$

Case S

$$\frac{\Gamma \vdash \mathbf{t} : T_i \ \forall i \in \{1, \dots, n\}}{\Gamma \vdash \left(\sum_{i=1}^{n} \alpha_i\right) \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot T_i} S$$

By the induction hypothesis, for all $i \in \{1, \dots, n\}$, $\Gamma[A/X] \vdash \mathbf{t} : T_i[A/X]$, so by rule S, $\Gamma[A/X] \vdash (\sum_{i=1}^{n} \alpha_i) \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot T_i[A/X] = (\sum_{i=1}^{n} \alpha_i \cdot T_i)[A/X]$.

Case $+_I$

$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R} +_I$$

By the induction hypothesis $\Gamma[A/X] \vdash \mathbf{t} : T[A/X]$ and $\Gamma[A/X] \vdash \mathbf{r} : R[A/X]$, so by rule $+_I$, $\Gamma[A/X] \vdash \mathbf{t} + \mathbf{r} : T[A/X] + R[A/X] = (T + R)[A/X]$.

Case $\equiv$

$$\frac{\Gamma \vdash \mathbf{t} : T \quad T \equiv R}{\Gamma \vdash \mathbf{t} : R} \equiv$$

By the induction hypothesis $\Gamma[A/X] \vdash \mathbf{t} : T[A/X]$, and since $T \equiv R$, then $T[A/X] \equiv R[A/X]$, so by rule $\equiv$, $\Gamma[A/X] \vdash \mathbf{t} : R[A/X]$.

Case 1_E

$$\frac{\Gamma \vdash 1 \cdot \mathbf{t} : T}{\Gamma \vdash \mathbf{t} : T} 1_E$$

By the induction hypothesis $\Gamma[A/X] \vdash 1 \cdot \mathbf{t} : T[A/X]$. By rule 1_E, $\Gamma[A/X] \vdash \mathbf{t} : T[A/X]$.

Item (2)

We proceed by induction on the typing derivation of $\Gamma, x : U \vdash \mathbf{t} : T$.

Case ax

$$\frac{}{\Gamma, x : U \vdash \mathbf{t} : T}\ ax$$

Cases:

- $\mathbf{t} = x$, then $T = U$, and so $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T$ and $\Gamma \vdash \mathbf{b} : U$ are the same sequent.
- $\mathbf{t} = y$. Notice that $y[\mathbf{b}/x] = y$. By Lemma Lemma 4.1.2 $\Gamma, x : U \vdash y : T$ implies $\Gamma \vdash y : T$.

Case $\to_I$

$$\frac{\Gamma, x : U, y : V \vdash \mathbf{r} : R}{\Gamma, x : U \vdash \lambda x.\lambda y.\mathbf{r} : V \to R}\ \to_I$$

Since our system admits weakening (Lemma 4.1.2), the sequent $\Gamma, y : V \vdash \mathbf{b} : U$ is derivable. Then by the induction hypothesis, $\Gamma, y : V \vdash \mathbf{r}[\mathbf{b}/x] : R$, from where, by rule $\to_I$, we obtain $\Gamma \vdash \lambda y.\mathbf{r}[\mathbf{b}/x] : V \to R$. We conclude, since $\lambda y.\mathbf{r}[\mathbf{b}/x] = (\lambda y.\mathbf{r})[\mathbf{b}/x]$.

Case $\to_E$

$$\frac{\Gamma, x : U \vdash \mathbf{r} : \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{Y}.(V \to T_i) \quad \Gamma, x : U \vdash \mathbf{u} : \sum_{j=1}^{m} \beta_j \cdot V[\vec{B}/\vec{Y}]}{\Gamma, x : U \vdash (\mathbf{r})\ \mathbf{u} : \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \times \beta_j \cdot R_i[\vec{B}/\vec{Y}]}\ \to_E$$

By the induction hypothesis, $\Gamma \vdash \mathbf{r}[\mathbf{b}/x] : \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{Y}.(V \to R_i)$ and $\Gamma \vdash \mathbf{u}[\mathbf{b}/x] : \sum_{j=1}^{m} \beta_j \cdot V[\vec{B}/\vec{Y}]$. Then, by rule $\to_E$, $\Gamma \vdash \mathbf{r}[\mathbf{b}/x]\ \mathbf{u}[\mathbf{b}/x] : \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \times \beta_j \cdot R_i[\vec{B}/\vec{Y}]$.

Case $\forall_I$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot V_i \quad Y \notin FV(\Gamma) \cup FV(U)}{\Gamma, x : U \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot V_i + \alpha_n \cdot \forall Y.V_n}\ \forall_I$$

By the induction hypothesis, $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : \sum_{i=1}^{n} \alpha_i \cdot V_i$. Then by rule $\forall_I$, $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : \sum_{i=1}^{n-1} \alpha_i \cdot V_i + \alpha_n \cdot \forall Y.V_n$.

Case $\forall_E$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot V_i + \alpha_n \cdot \forall Y.V_n}{\Gamma, x : U \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[B/Y]}\ \forall_E$$

By the induction hypothesis, $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : \sum_{i=1}^{n-1} \alpha_i \cdot V_i + \alpha_n \cdot \forall Y.V_n$. By rule $\forall_E$, $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : \sum_{i=1}^{n-1} \alpha_i \cdot V_i + \alpha_n \cdot V_n[B/Y]$.

Case S

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T_i \ \forall i \in \{1, \dots, n\}}{\Gamma, x : U \vdash \left(\sum_{i=1}^{n} \alpha_i\right) \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot T_i} S$$

By the induction hypothesis, for all $i \in \{1, \dots, n\}$, $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T_i$. Then by rule S, $\Gamma \vdash (\sum_{i=1}^{n} \alpha_i) \cdot \mathbf{t}[\mathbf{b}/x] : \sum_{i=1}^{n} \alpha_i \cdot T_i$. Notice that $(\sum_{i=1}^{n} \alpha_i) \cdot \mathbf{t}[\mathbf{b}/x] = ((\sum_{i=1}^{n} \alpha_i) \cdot \mathbf{t})[\mathbf{b}/x]$.

Case $+_I$

$$\frac{\Gamma, x : U \vdash \mathbf{r} : R \quad \Gamma, x : U \vdash \mathbf{u} : S}{\Gamma, x : U \vdash \mathbf{r} + \mathbf{u} : R + S} +_I$$

By the induction hypothesis, $\Gamma \vdash \mathbf{r}[\mathbf{b}/x] : R$ and $\Gamma \vdash \mathbf{u}[\mathbf{b}/x] : S$. Then by rule $+_I$, $\Gamma \vdash \mathbf{r}[\mathbf{b}/x] + \mathbf{u}[\mathbf{b}/x] : R + S$. Notice that $\mathbf{r}[\mathbf{b}/x] + \mathbf{u}[\mathbf{b}/x] = (\mathbf{r} + \mathbf{u})[\mathbf{b}/x]$.

Case $\equiv$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T \quad T \equiv R}{\Gamma, x : U \vdash \mathbf{t} : R} \equiv$$

By the induction hypothesis, $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : R$. Hence, by rule $\equiv$, $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T$.

Case 1_E

$$\frac{\Gamma, x : U \vdash 1 \cdot \mathbf{t} : T}{\Gamma, x : U \vdash \mathbf{t} : T} 1_E$$

By the induction hypothesis, $\Gamma \vdash 1 \cdot \mathbf{t}[\mathbf{b}/x] : R$. Hence, by rule 1_E, $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T$. □

We introduce the equivalence relation between contexts.

Definition 4.1.16. *The equivalence between contexts $\Gamma \equiv \Gamma'$ is defined by $x : A \in \Gamma$ if and only if there exists $x : A' \in \Gamma'$ such that $A \equiv A'$.*

4.2 Proof

We state the Subject Reduction theorem and prove that $\lambda^{\text{vec}*}$ satisfies it.

Theorem 4.2.1 (Subject Reduction)**.** *For any terms $\mathbf{t}, \mathbf{t}'$, any context Γ and any type T, if $\mathbf{t} \to \mathbf{t}'$ and $\Gamma \vdash \mathbf{t} : T$, then $\Gamma \vdash \mathbf{t}' : T$*

Proof. Let $\mathbf{t} \to \mathbf{t}'$ and $\Gamma \vdash \mathbf{t} : T$, we proceed by induction on the rewrite relation:

Group E

Case $1 \cdot \mathbf{t} \to \mathbf{t}$

Consider $\Gamma \vdash 1 \cdot \mathbf{t} : T$, then by 1_E rule, then $\Gamma \vdash \mathbf{t} : T$.

Case $\alpha \cdot (\beta \cdot \mathbf{t}) \to (\alpha \times \beta) \cdot \mathbf{t}$

Consider $\pi = \Gamma \vdash \alpha \cdot (\beta \cdot \mathbf{t}) : T$, then by applying Lemma 4.1.10, there exist $R_1, \dots, R_n$, $\alpha_1, \dots, \alpha_n$ such that

- $T \equiv \sum_{i=1}^{n} \alpha_i \cdot R_i$.
- $\pi_i = \Gamma \vdash \beta \cdot \mathbf{t} : R_i$, with $size(\pi) > size(\pi_i)$, for $i \in \{1, \dots, n\}$.
- $\sum_{i=1}^{n} \alpha_i = \alpha$.

By applying Lemma 4.1.10 for all $i \in \{1, \dots, n\}$, there exist $S_{(i,1)}, \dots, S_{(i,m_i)}$, $\beta_{(i,1)}, \dots, \beta_{(i,m_i)}$ such that

- $R_i \equiv \sum_{j=1}^{m_i} \beta_{(i,j)} \cdot S_{(i,j)}$.
- $\pi_{(i,j)} = \Gamma \vdash \mathbf{t} : S_{(i,j)}$, with $size(\pi_i) > size(\pi_{(i,j)})$, for $j \in \{1, \dots, m_i\}$.
- $\sum_{j=1}^{m_i} \beta_{(i,j)} = \beta$.

Notice that

$$\sum_{i=1}^{n} \alpha_i \cdot \underbrace{\sum_{j=1}^{m_i} \beta_{(i,j)}}_{\beta} = \sum_{i=1}^{n} \alpha_i \cdot \beta = \beta \cdot \underbrace{\sum_{i=1}^{n} \alpha_i}_{\alpha} = \beta \times \alpha = \alpha \times \beta$$

Then applying the S rule,

$$\frac{\Gamma \vdash \mathbf{t} : S_{(i,j)} \ \forall i \in \{1, \dots, n\},\ \forall j \in \{1, \dots, m_i\}}{\Gamma \vdash (\alpha \times \beta) \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot \sum_{j=1}^{m_i} \beta_{(i,j)} \cdot S_{(i,j)}} S$$

Since for all $i \in \{1, \dots, n\}$, $\sum_{j=1}^{m_i} \beta_{(i,j)} \cdot S_{(i,j)} \equiv R_i$, and since $\sum_{i=1}^{n} \alpha_i \cdot R_i \equiv T$, then by $\equiv$ rule, we conclude that $\Gamma \vdash (\alpha \times \beta) \cdot \mathbf{t} : T$.

Case $\alpha \cdot (\mathbf{t} + \mathbf{r}) \to \alpha \cdot \mathbf{t} + \alpha \cdot \mathbf{r}$

Consider $\Gamma \vdash \alpha \cdot (\mathbf{t} + \mathbf{r}) : T$, then by Lemma 4.1.10 there exist $R_1, \dots, R_n$, $\alpha_1, \dots, \alpha_n$ such that

- $T \equiv \sum_{i=1}^{n} \alpha_i \cdot R_i$.
- $\pi_i = \Gamma \vdash \mathbf{t} + \mathbf{r} : R_i$, with $size(\pi) > size(\pi_i)$, for $i \in \{1, \dots, n\}$.
- $\sum_{i=1}^{n} \alpha_i = \alpha$.

Since $size(\pi) > size(\pi_i)$, then by Lemma 4.1.11, for all $i \in \{1, \dots, n\}$, there exist $S_{i,1}, S_{i,2}$ such that

- $\Gamma \vdash \mathbf{t} : S_{(i,1)}$.
- $\Gamma \vdash \mathbf{r} : S_{(i,2)}$.
- $S_{(i,1)} + S_{(i,2)} \equiv R_i$.

Then applying the S rule,

$$\frac{\Gamma \vdash \mathbf{t} : S_{(i,1)} \ \forall i \in \{1, \ldots, n\}}{\Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot S_{(i,1)}} S \qquad \frac{\Gamma \vdash \mathbf{r} : S_{(i,2)} \ \forall i \in \{1, \ldots, n\}}{\Gamma \vdash \alpha \cdot \mathbf{r} : \sum_{i=1}^{n} \alpha_i \cdot S_{(i,2)}} S$$

By applying the $+_I$ rule,

$$\frac{\Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot S_{(i,1)} \quad \Gamma \vdash \alpha \cdot \mathbf{r} : \sum_{i=1}^{n} \alpha_i \cdot S_{(i,2)}}{\Gamma \vdash \alpha \cdot \mathbf{t} + \alpha \cdot \mathbf{r} : \sum_{i=1}^{n} \alpha_i \cdot S_{(i,1)} + \sum_{i=1}^{n} \alpha_i \cdot S_{(i,2)}} +_I$$

Notice that

$$\sum_{i=1}^{n} \alpha_i \cdot S_{(i,1)} + \sum_{i=1}^{n} \alpha_i \cdot S_{(i,2)} \equiv \sum_{i=1}^{n} \alpha_i \cdot (S_{(i,1)} + S_{(i,2)}) \equiv \sum_{i=1}^{n} \alpha_i \cdot R_i \equiv T$$

Finally, applying the $\equiv$ rule, we conclude that $\Gamma \vdash \alpha \cdot \mathbf{t} + \alpha \cdot \mathbf{r} : T$.

Group F

Case $\alpha \cdot \mathbf{t} + \beta \cdot \mathbf{t} \to (\alpha + \beta) \cdot \mathbf{t}$

Consider $\Gamma \vdash \alpha \cdot \mathbf{t} + \beta \cdot \mathbf{t} : T$.
For simplicity, we rename $\alpha = \mu_1$ and $\beta = \mu_2$, then by Lemma 4.1.11 there exist S_1, S_2 such that

- $\pi_1 = \Gamma \vdash \mu_1 \cdot \mathbf{t} : S_1$.
- $\pi_2 = \Gamma \vdash \mu_2 \cdot \mathbf{t} : S_2$.
- $S_1 + S_2 \equiv T$.

And by Lemma 4.1.10, for $k = 1, 2$, there exist $R_{(k,1)}, \ldots, R_{(k,n_k)}$, $\gamma_{(k,1)}, \ldots, \gamma_{(k,n_k)}$ such that

- $S_k \equiv \sum_{i=1}^{n_k} \gamma_{(k,i)} \cdot R_{(k,i)}$.
- $\pi_{(k,i)} = \Gamma \vdash \mathbf{t} : R_{(k,i)}$, with $size(\pi_k) > size(\pi_{(k,i)})$, for $i \in \{1, \ldots, n_k\}$.
- $\sum_{i=1}^{n_k} \gamma_{(k,i)} = \mu_k$.

Notice that

$$\underbrace{\sum_{i=1}^{n_1} \mu_{(1,i)}}_{=\,\mu_1} + \underbrace{\sum_{i=1}^{n_2} \mu_{(2,i)}}_{=\,\mu_2} = \mu_1 + \mu_2 = \alpha + \beta$$

Then applying the S rule,

$$\frac{\Gamma \vdash \mathbf{t} : R_{(1,i)} \ \forall i \in \{1, \ldots, n_1\} \quad \Gamma \vdash \mathbf{t} : R_{(2,i)} \ \forall i \in \{1, \ldots, n_2\}}{\Gamma \vdash (\alpha + \beta) \cdot \mathbf{t} : \sum_{i=1}^{n_1} \mu_{(1,i)} \cdot R_{(1,i)} + \sum_{i=1}^{n_2} \mu_{(2,i)} \cdot R_{(2,i)}} S$$

We also know that

$$\sum_{i=1}^{n_1} \mu_{(1,i)} \cdot R_{(1,i)} \equiv S_1 \qquad \sum_{i=1}^{n_2} \mu_{(2,i)} \cdot R_{(2,i)} \equiv S_2 \qquad S_1 + S_2 \equiv T$$

Finally, we conclude by $\equiv$ rule that $\Gamma \vdash (\alpha+\beta)\cdot \mathbf{t} : T$.

Case $\alpha \cdot \mathbf{t} + \mathbf{t} \to (\alpha+1)\cdot \mathbf{t}$

Consider $\Gamma \vdash \alpha \cdot \mathbf{t} + \mathbf{t} : T$, then by Lemma 4.1.11 there exist S_1, S_2 such that

- $\pi = \Gamma \vdash \alpha \cdot \mathbf{t} : S_1$.
- $\Gamma \vdash \mathbf{t} : S_2$.
- $S_1 + S_2 \equiv T$.

And by Lemma 4.1.10, there exist $R_1, \dots, R_n$, $\alpha_1, \dots, \alpha_n$ such that

- $S_1 \equiv \sum_{i=1}^{n} \alpha_i \cdot R_i$.
- $\pi_i = \Gamma \vdash \mathbf{t} : R_i$, with $size(\pi) > size(\pi_i)$, for $i \in \{1, \dots, n\}$.
- $\sum_{i=1}^{n} \alpha_i = \alpha$.

Then applying the S rule,

$$\dfrac{\Gamma \vdash \mathbf{t} : R_i \ \forall i \in \{1,\dots,n\} \quad \Gamma \vdash \mathbf{t} : S_2}{\Gamma \vdash (\alpha+1)\cdot \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot R_i + S_2}\, S$$

We also know that

$$\sum_{i=1}^{n} \mu_i \cdot R_i \equiv S_1 \qquad S_1 + S_2 \equiv T$$

Finally, we conclude by $\equiv$ rule that $\Gamma \vdash (\alpha+1)\cdot \mathbf{t} : T$.

Case $\mathbf{t} + \mathbf{t} \to (1+1)\cdot \mathbf{t}$

Consider $\Gamma \vdash \mathbf{t} + \mathbf{t} : T$, then by Lemma 4.1.11 there exist T_1, T_2 such that

- $\Gamma \vdash \mathbf{t} : T_1$.
- $\Gamma \vdash \mathbf{t} : T_2$.
- $T_1 + T_2 \equiv T$.

Then applying the S rule,

$$\dfrac{\Gamma \vdash \mathbf{t} : T_1 \quad \Gamma \vdash \mathbf{t} : T_2}{\Gamma \vdash (1+1)\cdot \mathbf{t} : T_1 + T_2}\, S$$

Finally, by $\equiv$ rule we conclude that $\Gamma \vdash (1+1)\cdot \mathbf{t} : T$.

Group B

Case $(\lambda x.\mathbf{t})\ \mathbf{b} \to \mathbf{t}[\mathbf{b}/x]$

Consider $\Gamma \vdash (\lambda x.\mathbf{t})\ \mathbf{b} : T$, then by Lemma 4.1.12, , there exist $R_1, \dots, R_h$, $\mu_1, \dots, \mu_h$, $\mathcal{V}_1, \dots, \mathcal{V}_h$ such that $T \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \dots, h\}$,

- $\Gamma \vdash \lambda x.\mathbf{t} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\Gamma \vdash \mathbf{b} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_{(k,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$.

For the sake of readability, we will split the proof:

1. We will prove that $\Gamma, x : U[\vec{A}_{(k,j)}/X] \vdash \mathbf{t} : T_{(k,i)}[\vec{A}_{(k,j)}/X]$, for all $k \in \{1, \ldots, h\}$, $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$.
2. We will prove that $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T_{(k,i)}[\vec{A}_{(k,j)}/X]$, for all $k \in \{1, \ldots, h\}$, $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$.
3. We will prove that $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T$.

Item (1) ....

We will prove that $\Gamma, x : U[\vec{A}_{(k,j)}/X] \vdash \mathbf{t} : T_{(k,i)}[\vec{A}_{(k,j)}/X]$, for all $k \in \{1, \ldots, h\}$, $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$.
For simplicity, we will omit the k index, which would otherwise be present in all the types, scalars and upper bound of the summations.
Considering $\lambda x.\mathbf{t}$ is a basis term, by Lemma 4.1.14 then there exist $W_1, \ldots, W_b$, $\gamma_1, \ldots, \gamma_b$ such that

- $\sum_{a=1}^{b} \gamma_a \cdot W_a \equiv \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i)$.
- $\Gamma \vdash \lambda x.\mathbf{t} : W_a$, for $a \in \{1, \ldots, b\}$.
- $\sum_{a=1}^{b} \gamma_a = 1$.

Without loss of generality, we assume that all unit types present at both sides of the equivalences are distinct, so by Lemma 4.1.3, then $b = n$ and there exists a permutation of n, p, such that $\forall \vec{X}.(U \to T_i) \equiv W_{p(i)}$ and $\alpha_i = \gamma_{p(i)}$, for all $i \in \{1, \ldots, n\}$.
Since for all $i \in \{1, \ldots, n\}$ we have $\Gamma \vdash \lambda x.\mathbf{t} : \forall \vec{X}.(U \to T_i)$, then by Lemma 4.1.13 and Lemma 4.1.3, we know that $\Gamma, x : V_i \vdash \mathbf{t} : S_i$, and $V_i \to S_i \preceq_{\mathcal{V}_i,\Gamma} \forall \vec{X}.(U \to T_i)$.
By applying Lemma 4.1.9, then $U \equiv V_i[\vec{B}/\vec{Y}]$ and $T_i \equiv S_i[\vec{B}/\vec{Y}]$, with $\vec{Y} \notin FV(\Gamma)$.
Then, by Lemma 4.1.15 and $\equiv$ rule, we have that $\Gamma, x : U \vdash \mathbf{t} : T_i$ for all $i \in \{1, \ldots, n\}$.
By Lemma 4.1.8, since $V_i \to S_i \preceq_{\mathcal{V}_i,\Gamma} \forall \vec{X}.(U \to T_i)$ for all $i \in \{1, \ldots, n\}$, then we know $\vec{X} \notin FV(\Gamma)$ and so by Definition 4.1.16, $\Gamma \equiv \Gamma[\vec{C}/\vec{X}]$, for any $\vec{C}$.
Therefore, by applying Lemma 4.1.15 multiple times, we have $\Gamma, x : U[\vec{A}_j/X] \vdash \mathbf{t} : T_i[\vec{A}_j/X]$ for all $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$.
Following this procedure for all $k \in \{1, \ldots, h\}$, then we proved that $\Gamma, x : U[\vec{A}_{(k,j)}/X] \vdash \mathbf{t} : T_{(k,i)}[\vec{A}_{(k,j)}/X]$, for all $k \in \{1, \ldots, h\}$, $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$.

Item (2) ....

We will prove that $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T_{(k,i)}[\vec{A}_{(k,j)}/X]$, for all $k \in \{1, \ldots, h\}$, $j \in \{1, \ldots, m_k\}$, $i \in \{1, \ldots, n_k\}$.
For simplicity, we will omit the k index, which would otherwise be present in all the types, scalars and upper bound of the summations.
Since $\mathbf{b}$ is a basis term, by Lemma 4.1.14 there exist $W'_1, \ldots, W'_c$, $\eta_1, \ldots, \eta_c$ such that

- $\sum_{a=1}^{c} \eta_a \cdot W'_a \equiv \sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j/\vec{X}]$.
- $\Gamma \vdash \mathbf{b} : W'_a$, for $a \in \{1, \ldots, c\}$.
- $\sum_{a=1}^{c} \eta_a = 1$.

Without loss of generality, we assume that all unit types present at both sides of the equivalences are distinct, so by Lemma 4.1.3, then $c = m$, and there exists a permutation q of m, such that $U[\vec{A}_j/\vec{X}] \equiv W'_{q(j)}$ and $\beta_j = \eta_{q(j)}$, for all $j \in \{1, \dots, m\}$.
Then, following Item (1), by applying Lemma 4.1.15, we have that $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T_i[\vec{A}_j/X]$ for all $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$. Following this procedure for all $k \in \{1, \dots, h\}$, then we proved that $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T_{(k,i)}[\vec{A}_{(k,j)}/X]$, for all $k \in \{1, \dots, h\}$, $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$.

Item (3) ....

Using the results of Item (1) and Item (2), and since in both items we already proved that for all $k \in \{1, \dots, h\}$, $\sum_{i=1}^{n_k} \alpha_i = \sum_{j=1}^{m_k} \beta_j = 1$, then by applying the S rule for all $k \in \{1, \dots, h\}$ (we will omit the k index for simplicity, that will be present in all types, scalars and upper bound of the summations),

$$\cfrac{\cfrac{\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T_i[\vec{A}_j/X] \ \forall i \in \{1, \dots, n\},\ \forall j \in \{1, \dots, m\}}{\Gamma \vdash 1 \cdot \mathbf{t}[\mathbf{b}/x] : \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/X]}\ S}{\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/X]}\ 1_E$$

Since $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/X] \preceq_{\mathcal{V},\Gamma} R_k$, then $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : R_k$.
Considering that $\sum_{k=1}^{h} \mu_k = 1$, then by applying the S and the 1_E rule again,

$$\cfrac{\cfrac{\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : R_k \ \forall k \in \{1, \dots, h\}}{\Gamma \vdash 1 \cdot \mathbf{t}[\mathbf{b}/x] : \sum_{k=1}^{h} \mu_k \cdot R_k}\ S}{\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : \sum_{k=1}^{h} \mu_k \cdot R_k}\ 1_E$$

Finally, since $\mu_k \cdot R_k \equiv T$, we conclude by $\equiv$ rule that $\Gamma \vdash \mathbf{t}[\mathbf{b}/x] : T$.

.... **Group A** ....

.... **Case $(\mathbf{t} + \mathbf{r})\ \mathbf{u} \to (\mathbf{t})\ \mathbf{u} + (\mathbf{r})\ \mathbf{u}$** ....

Consider $\Gamma \vdash (\mathbf{t} + \mathbf{r})\ \mathbf{u} : T$, then by Lemma 4.1.12, there exist $R_1, \dots, R_h$, $\mu_1, \dots, \mu_h$, $\mathcal{V}_1, \dots, \mathcal{V}_h$ such that $T \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \dots, h\}$

- $\Gamma \vdash \mathbf{t} + \mathbf{r} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\Gamma \vdash \mathbf{u} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_j/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$.

We will simplify the rest of this proof by omitting the k index, which would otherwise be present in all the types, scalars and upper bound of the summations. The rest of this proof then should be applied to all $k \in \{1, \dots, h\}$.
By Lemma 4.1.11, there exist S_1, S_2 such that

- $\Gamma \vdash \mathbf{t} : S_1$.

- $\Gamma \vdash \mathbf{r} : S_2$.
- $S_1 + S_2 \equiv \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i)$.

Hence, there exist $N_1, N_2 \subseteq \{1, \ldots, n\}$ with $N_1 \cup N_2 = \{1, \ldots, n\}$ such that

$$S_1 \equiv \sum_{i \in N_1 \setminus N_2} \alpha_i \cdot \forall \vec{X}.(U \to T_i) + \sum_{i \in N_1 \cap N_2} \eta_i \cdot \forall \vec{X}.(U \to T_i) \quad \text{and}$$
$$S_2 \equiv \sum_{i \in N_2 \setminus N_1} \alpha_i \cdot \forall \vec{X}.(U \to T_i) + \sum_{i \in N_1 \cap N_2} \eta'_i \cdot \forall \vec{X}.(U \to T_i)$$

where for all $i \in N_1 \cap N_2$, $\eta_i + \eta'_i = \alpha_i$. Therefore, using $\equiv$ we get

$$\Gamma \vdash \mathbf{t} : \sum_{i \in N_1 \setminus N_2} \alpha_i \cdot \forall \vec{X}.(U \to T_i) + \sum_{i \in N_1 \cap N_2} \eta_i \cdot \forall \vec{X}.(U \to T_i) \quad \text{and}$$
$$\Gamma \vdash \mathbf{r} : \sum_{i \in N_2 \setminus N_1} \alpha_i \cdot \forall \vec{X}.(U \to T_i) + \sum_{i \in N_1 \cap N_2} \eta'_i \cdot \forall \vec{X}.(U \to T_i)$$

So, using rule $\to_E$, we get

$$\Gamma \vdash (\mathbf{t})\ \mathbf{u} : \sum_{i \in N_1 \setminus N_2} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}] + \sum_{i \in N_1 \cap N_2} \sum_{j=1}^{m} \eta'_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}] \quad \text{and}$$
$$\Gamma \vdash (\mathbf{r})\ \mathbf{u} : \sum_{i \in N_2 \setminus N_1} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}] + \sum_{i \in N_1 \cap N_2} \sum_{j=1}^{m} \eta'_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}]$$

By rule $+_I$ we can conclude

$$\Gamma \vdash (\mathbf{t})\ \mathbf{u} + (\mathbf{r})\ \mathbf{u} : \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}]$$

Since $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$ for all $k \in \{1, \ldots, h\}$, then by definition of $\preceq$, we can derive $\Gamma \vdash (\mathbf{t})\ \mathbf{u} + (\mathbf{r})\ \mathbf{u} : R_k$.
By applying the S and 1_E rules, then

$$\dfrac{\dfrac{\Gamma \vdash (\mathbf{t})\ \mathbf{u} + (\mathbf{r})\ \mathbf{u} : R_k\ \forall k \in \{1, \ldots, k\}}{\Gamma \vdash 1 \cdot ((\mathbf{t})\ \mathbf{u} + (\mathbf{r})\ \mathbf{u}) : \sum_{k=1}^{h} \mu_k \cdot R_k}\ S}{\Gamma \vdash (\mathbf{t})\ \mathbf{u} + (\mathbf{r})\ \mathbf{u} : \sum_{k=1}^{h} \mu_k \cdot R_k}\ 1_E$$

Finally, by the $\equiv$ rules, then $\Gamma \vdash (\mathbf{t})\ \mathbf{u} + (\mathbf{r})\ \mathbf{u} : T$.

Case $(\mathbf{t})\ (\mathbf{r} + \mathbf{u}) \to (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{u}$

Consider $\Gamma \vdash (\mathbf{t})\ (\mathbf{r} + \mathbf{u}) : T$, then by Lemma 4.1.12, there exist $R_1, \ldots, R_h$, $\mu_1, \ldots, \mu_h$, $\mathcal{V}_1, \ldots, \mathcal{V}_h$ such that $T \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \ldots, h\}$

- $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\Gamma \vdash \mathbf{r} + \mathbf{u} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_{(k,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$.

We will simplify the rest of this proof by omitting the k index, which would otherwise be present in all the types, scalars and upper bound of the summations. The rest of this proof then should be applied to all $k \in \{1, \dots, h\}$.
By Lemma 4.1.11, there exists S_1, S_2 such that

- $\Gamma \vdash \mathbf{r} : S_1$
- $\Gamma \vdash \mathbf{u} : S_2$
- $S_1 + S_2 \equiv \sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j/\vec{X}]$

Hence, there exist $N_1, N_2 \subseteq \{1, \dots, m\}$ with $N_1 \cup N_2 = \{1, \dots, m\}$, such that

$$S_1 \equiv \sum_{j \in N_1 \setminus N_2} \beta_j \cdot U[\vec{A}_j/\vec{X}] + \sum_{j \in N_1 \cap N_2} \eta_{kj} \cdot U[\vec{A}_j/\vec{X}] \quad \text{and}$$
$$S_2 \equiv \sum_{i \in N_2 \setminus N_1} \beta_j \cdot U[\vec{A}_j/\vec{X}] + \sum_{j \in N_1 \cap N_2} \eta'_{kj} \cdot U[\vec{A}_j/\vec{X}]$$

where for all $j \in N_1 \cap N_2$, $\eta_{kj} + \eta'_{kj} = \beta_j$. Therefore, using $\equiv$ we get

$$\Gamma \vdash \mathbf{r} : \sum_{j \in N_1 \setminus N_2} \beta_j \cdot U[\vec{A}_j/\vec{X}] + \sum_{j \in N_1 \cap N_2} \eta_{kj} \cdot U[\vec{A}_j/\vec{X}] \quad \text{and}$$
$$\Gamma \vdash \mathbf{u} : \sum_{j \in N_2 \setminus N_1} \beta_j \cdot U[\vec{A}_j/\vec{X}] + \sum_{j \in N_1 \cap N_2} \eta'_{kj} \cdot U[\vec{A}_j/\vec{X}]$$

So, using rule $\to_E$, we get

$$\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n} \sum_{j \in N_1 \setminus N_2} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}] + \sum_{i=1}^{n} \sum_{j \in N_1 \cap N_2} \alpha_i \times \eta_{kj} \cdot T_i[\vec{A}_j/\vec{X}] \quad \text{and}$$
$$\Gamma \vdash (\mathbf{t})\ \mathbf{u} : \sum_{i=1}^{n} \sum_{j \in N_2 \setminus N_1} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}] + \sum_{i=1}^{n} \sum_{j \in N_1 \cap N_2} \alpha_i \times \eta'_{kj} \cdot T_i[\vec{A}_j/\vec{X}]$$

By rule $+_I$ we can conclude

$$\Gamma \vdash (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{u} : \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \times \beta_j \cdot T_i[\vec{A}_j/\vec{X}]$$

Since $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$ for all $k \in \{1, \dots, h\}$, then by definition of $\preceq$, we can derive $\Gamma \vdash (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{u} : R_k$.
By applying the S and 1_E rules, then

$$\cfrac{\cfrac{\Gamma \vdash (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{u} : R_k\ \forall k \in \{1, \dots, h\}}{\Gamma \vdash 1 \cdot ((\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{u}) : \sum_{k=1}^{h} \mu_k \cdot R_k}\, S}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{u} : \sum_{k=1}^{h} \mu_k \cdot R_k}\, 1_E$$

Finally, by the $\equiv$ rules, then $\Gamma \vdash (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{u} : T$.

Case $(\alpha \cdot \mathbf{t})\ \mathbf{r} \to \alpha \cdot (\mathbf{t})\ \mathbf{r}$

Consider $\Gamma \vdash (\alpha \cdot \mathbf{t})\ \mathbf{r} : T$, by Lemma 4.1.12, there exist $R_1, \dots, R_h$, $\mu_1, \dots, \mu_h$, $\mathcal{V}_1, \dots, \mathcal{V}_h$ such that $T \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \dots, h\}$

- $\pi_k = \Gamma \vdash \alpha \cdot \mathbf{t} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\Gamma \vdash \mathbf{r} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_{(k,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$.

We will simplify the rest of this proof by omitting the k index, which would otherwise be present in all the types, scalars and upper bound of the summations. The rest of this proof then should be applied to all $k \in \{1, \dots, h\}$.
By Lemma 4.1.10, there exist $S_1, \dots, S_b$, $\eta_1, \dots, \eta_b$ such that

- $\sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i) \equiv \sum_{a=1}^{b} \eta_a \cdot S_a$.
- $\pi_i = \Gamma \vdash \mathbf{t} : S_a$, with $size(\pi) > size(\pi_a)$, for $a \in \{1, \dots, b\}$.
- $\sum_{a=1}^{b} \eta_a = \alpha$.

Considering $\sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i)$ does not have any general variable $\mathbb{X}$ and that $\sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i) \equiv \sum_{a=1}^{b} \eta_a \cdot S_a$, then by Lemma 4.1.1, $S_a \equiv \sum_{c=1}^{d_a} \gamma_{(a,c)} \cdot V_{(a,c)}$.
Without loss of generality, we assume that all unit types present at both sides of the equivalences are distinct, so by Lemma 4.1.3, then $n = \sum_{a=1}^{b} d_a$, and by taking a partition from $\{1, \dots, \sum_{a=1}^{b} d_a\}$ (defining an equivalence class) and the trivial permutation p of n such that $p(i) = i$ (which we will omit for readability), we have

- $\alpha_i = \eta_{[i]} \times \sigma_i$, where $\sigma_i = \gamma_{\left([i], \frac{i}{[i]}\right)}$.
- $\forall \vec{X}.(U \to T_i) \equiv V_{\left([i], \frac{i}{[i]}\right)}$.

Take $f(a) = \sum_{e=1}^{a-1} d_e$, so we rewrite $S_a \equiv \sum_{c=1}^{d_a} \gamma_{(a,c)} \cdot V_{(a,c)}$ as

$$S_a \equiv \sum_{g=f(a)}^{f(a)+d_a} \sigma_g \cdot V_{\left([g], \frac{g}{[g]}\right)} \equiv \sum_{g=f(a)}^{f(a)+d_a} \sigma_g \cdot \forall \vec{X}.(U \to T_g)$$

Applying $\to_E$ for all $a \in \{1, \dots, b\}$,

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{g=f(a)}^{f(a)+d_a} \sigma_g \cdot \forall \vec{X}.(U \to T_g) \quad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j/\vec{X}]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{g=f(a)}^{f(a)+d_a} \sum_{j=1}^{m} (\sigma_g \times \beta_j) \cdot T_g[\vec{A}_j/\vec{X}]} \to_E$$

We rewrite $\sum_{g=f(a)}^{f(a)+d_a} \sum_{j=1}^{m} (\sigma_g \times \beta_j) \cdot T_g[\vec{A}_j/\vec{X}] \equiv P_a$, then by applying the S rule we have

$$\frac{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : P_a \ \forall a \in \{1, \dots, b\}}{\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : \sum_{a=1}^{b} \eta_a \cdot P_a} S$$

Now we begin to unravel the final result

$$\sum_{a=1}^{b} \eta_a \cdot P_a \equiv \sum_{a=1}^{b} \eta_a \cdot \sum_{g=f(a)}^{f(a)+d_a} \sum_{j=1}^{m} (\sigma_g \times \beta_j) \cdot T_g[\vec{A}_j/\vec{X}]$$

$$\equiv \sum_{a=1}^{b} \sum_{g=f(a)}^{f(a)+d_a} \sum_{j=1}^{m} \left(\eta_{[g]} \times \sigma_g \times \beta_j\right) \cdot T_g[\vec{A}_j/\vec{X}]$$

$$\equiv \sum_{a=1}^{b} \sum_{g=f(a)}^{f(a)+d_a} \sum_{j=1}^{m} (\alpha_g \times \beta_j) \cdot T_g[\vec{A}_j/\vec{X}]$$

$$\equiv \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \times \beta_j) \cdot T_i[\vec{A}_j/\vec{X}]$$

Then,

$$\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \times \beta_j) \cdot T_i[\vec{A}_j/\vec{X}]$$

Since $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} (\alpha_{(k,i)} \times \beta_{(k,j)}) \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$, then for all $k \in \{1, \ldots, h\}$, $\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : R_k$.
By applying the S and 1_E rules, then

$$\dfrac{\dfrac{\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : R_k\ \forall k \in \{1, \ldots, h\}}{\Gamma \vdash 1 \cdot (\alpha \cdot (\mathbf{t})\ \mathbf{r}) : \sum_{k=1}^{h} \mu_k \cdot R_k}\ S}{\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : \sum_{k=1}^{h} \mu_k \cdot R_k}\ 1_E$$

Finally, by the $\equiv$ rule, then $\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : T$. □

Case $(\mathbf{t})\ (\alpha \cdot \mathbf{r}) \to \alpha \cdot (\mathbf{t})\ \mathbf{r}$

Consider $\Gamma \vdash (\mathbf{t})\ (\alpha \cdot \mathbf{r}) : T$, by Lemma 4.1.12, there exist $R_1, \ldots, R_h$, $\mu_1, \ldots, \mu_h$, $\mathcal{V}_1, \ldots, \mathcal{V}_h$ such that $T \equiv \sum_{k=1}^{h} \mu_k \cdot R_k$, $\sum_{k=1}^{h} \mu_k = 1$ and for all $k \in \{1, \ldots, h\}$

- $\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n_k} \alpha_{(k,i)} \cdot \forall \vec{X}.(U \to T_{(k,i)})$.
- $\pi_k = \Gamma \vdash \alpha \cdot \mathbf{r} : \sum_{j=1}^{m_k} \beta_{(k,j)} \cdot U[\vec{A}_{(k,j)}/\vec{X}]$.
- $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} \alpha_{(k,i)} \times \beta_{(k,j)} \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$.

We will simplify the rest of this proof by omitting the k index, which would otherwise be present in all the types, scalars and upper bound of the summations. The rest of this proof then should be applied to all $k \in \{1, \ldots, h\}$.
By Lemma 4.1.10, there exist $S_1, \ldots, S_b$, $\eta_1, \ldots, \eta_b$ such that

- $\sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j/\vec{X}] \equiv \sum_{a=1}^{b} \eta_a \cdot S_a$.
- $\pi_i = \Gamma \vdash \mathbf{r} : S_a$, with $size(\pi) > size(\pi_a)$, for $a \in \{1, \ldots, b\}$.
- $\sum_{a=1}^{b} \eta_a = \alpha$.

Considering $\sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j/\vec{X}]$ does not have any general variable $\mathbb{X}$ and that $\sum_{j=1}^{m} \beta_j \cdot U[\vec{A}_j/\vec{X}] \equiv \sum_{a=1}^{b} \eta_a \cdot S_a$, then by Lemma 4.1.1, $S_a \equiv \sum_{c=1}^{d_a} \gamma_{(a,c)} \cdot V_{(a,c)}$.
Without loss of generality, we assume that all unit types present at both sides of the equivalences are distinct, so by Lemma 4.1.3, then $m = \sum_{a=1}^{b} d_a$, and by taking a partition from $\{1, \ldots, \sum_{a=1}^{b} d_a\}$ (defining an equivalence class) and the trivial permutation p of m such that $p(j) = j$ (which we will omit for readability), we have

- $\beta_j = \eta_{[j]} \times \sigma_j$, where $\sigma_j = \gamma_{\left([j], \frac{j}{[j]}\right)}$.
- $U[\vec{A}_j/\vec{X}] \equiv V_{\left([j], \frac{j}{[j]}\right)}$.

Take $f(a) = \sum_{e=1}^{a-1} d_e$, so we rewrite $S_a \equiv \sum_{c=1}^{d_a} \gamma_{(a,c)} \cdot V_{(a,c)}$ as

$$S_a \equiv \sum_{g=f(a)}^{f(a)+d_a} \sigma_g \cdot V_{\left([g], \frac{g}{[g]}\right)} \equiv \sum_{g=f(a)}^{f(a)+d_a} \sigma_g \cdot U[\vec{A}_g/\vec{X}]$$

Applying $\to_E$ for all $a \in \{1, \dots, b\}$,

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot \forall \vec{X}.(U \to T_i) \quad \Gamma \vdash \mathbf{r} : \sum_{g=f(a)}^{f(a)+d_a} \sigma_g \cdot U[\vec{A}_g/\vec{X}]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n} \sum_{g=f(a)}^{f(a)+d_a} (\alpha_i \times \sigma_g) \cdot T_i[\vec{A}_g/\vec{X}]} \to_E$$

We rewrite $\sum_{i=1}^{n} \sum_{g=f(a)}^{f(a)+d_a} (\alpha_i \times \sigma_g) \cdot T_i[\vec{A}_g/\vec{X}] \equiv P_a$, then by applying the S rule we have

$$\frac{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : P_a \ \forall a \in \{1, \dots, b\}}{\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : \sum_{a=1}^{b} \eta_a \cdot P_a} S$$

Now we begin to unravel the final result

$$\begin{aligned}
\sum_{a=1}^{b} \eta_a \cdot P_a &\equiv \sum_{a=1}^{b} \eta_a \cdot \sum_{i=1}^{n} \sum_{g=f(a)}^{f(a)+d_a} (\alpha_i \times \sigma_g) \cdot T_i[\vec{A}_g/\vec{X}] \\
&\equiv \sum_{a=1}^{b} \sum_{g=f(a)}^{f(a)+d_a} \sum_{j=1}^{m} \left(\alpha_i \times \eta_{[g]} \times \sigma_g\right) \cdot T_i[\vec{A}_g/\vec{X}] \\
&\equiv \sum_{a=1}^{b} \sum_{g=f(a)}^{f(a)+d_a} \sum_{j=1}^{m} (\alpha_i \times \beta_g) \cdot T_i[\vec{A}_g/\vec{X}] \\
&\equiv \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \times \beta_j) \cdot T_i[\vec{A}_j/\vec{X}]
\end{aligned}$$

Then,

$$\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \times \beta_j) \cdot T_i[\vec{A}_j/\vec{X}]$$

Since $\sum_{i=1}^{n_k} \sum_{j=1}^{m_k} (\alpha_{(k,i)} \times \beta_{(k,j)}) \cdot T_{(k,i)}[\vec{A}_{(k,j)}/\vec{X}] \preceq_{\mathcal{V}_k,\Gamma} R_k$, then for all $k \in \{1, \dots, h\}$, $\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : R_k$.

By applying the S and 1_E rules, then

$$\dfrac{\dfrac{\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : R_k\ \forall k \in \{1, \ldots, h\}}{\Gamma \vdash 1 \cdot (\alpha \cdot (\mathbf{t})\ \mathbf{r}) : \sum_{k=1}^{h} \mu_k \cdot R_k}\ S}{\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : \sum_{k=1}^{h} \mu_k \cdot R_k}\ 1_E$$

Finally, by the $\equiv$ rule, then $\Gamma \vdash \alpha \cdot (\mathbf{t})\ \mathbf{r} : T$.

Chapter 5

Other properties

Chapter Summary

We present the Progress property and we prove it is satisfied by $\lambda^{\text{vec}*}$. We also formalise the concept of weight of terms and types, and prove that the weight of normalized terms is the same as the weight of their types. ■

WE present two additional properties that are satisfied by $\lambda^{\text{vec}*}$. One of them is the property of Progress, satisfied by λ^{vec} [3, Theorem 6.1], which allows us to characterise the form of the values. We prove that $\lambda^{\text{vec}*}$ also satisfies this property.

In this chapter we also formalise the concept of weight for both types and terms, which refers to the sum of all the components of the vectors they model. From this definition we formulate a new property, Weight Preservation, which guarantees that for any typed term $\vdash \mathbf{t} : T$, upon normalizing $\mathbf{t} \to^* \mathbf{v}$, the weight of $\mathbf{v}$ is the same as the weight of T. This result provides a way to statically characterise the weight of a term before it is reduced, by looking at the weight of its type.

Plan of this chapter. In Section 5.1 we prove the Progress property is satisfied by $\lambda^{\text{vec}*}$. In Section 5.2 we formalise the concept of weight of terms and types, and we state and prove the property of Weight Preservation.

5.1 Progress

We state and prove the Progress property.

Theorem 5.1.1 (Progress)**.** *Given* $\mathbb{V} = \left\{\sum_{i=1}^{n} \alpha_i \cdot \mathbf{b}_i + \sum_{j=n+1}^{m} \mathbf{b}_j \mid \forall i, j, \mathbf{b}_i \neq \mathbf{b}_j\right\}$ *and* NF *the set of terms in normal form (the terms that cannot be reduced any further), then if* $\vdash \mathbf{t} : T$ *and* $\mathbf{t} \in \mathsf{NF}$*, it follows that* $\mathbf{t} \in \mathbb{V}$*.*

Proof. By induction on $\mathbf{t}$:

Case $\mathbf{t} = \sum_{i=1}^{n} \alpha_i \cdot \mathbf{b}_i + \sum_{j=n+1}^{m} \mathbf{b}_j \mid \forall i, j, \mathbf{b}_i \neq \mathbf{b}_j$

Trivial case.

Case $\mathbf{t} = \sum_{i=1}^{n} \alpha_i \cdot \mathbf{b}_i + \sum_{j=n+1}^{m} \mathbf{b}_j \mid \exists i, j, \mathbf{b}_i = \mathbf{b}_j$

$\mathbf{t} \notin \mathsf{NF}$, since at least one reduction rule from Group F can be applied.

Case $\mathbf{t} = (\mathbf{r})\ \mathbf{s}$

By induction hypothesis, we know that $\mathbf{r} = \sum_{i=1}^{n} \alpha_i \cdot \mathbf{b}_i + \sum_{j=n+1}^{m} \mathbf{b}_j \in \mathbb{V}$. We consider the following cases:

- If $m > n+1$ or $n \neq 0$, then at least one reduction rule from Group A can be applied, hence $(\mathbf{r})\ \mathbf{s} \notin \mathsf{NF}$.
- If $m = n+1$ and $n = 0$, then $\mathbf{r} = \mathbf{b}_{n+1} \in \mathbb{V}$. Since $FV(\mathbf{r}) = \emptyset$, then $\mathbf{r} = \lambda x.\mathbf{r}'$, which implies $(\mathbf{r})\ \mathbf{s}$ is a beta-redex or at least one reduction rule from Group A can be applied, hence $(\mathbf{r})\ \mathbf{s} \notin \mathsf{NF}$.

Case $\mathbf{t} = \alpha \cdot \mathbf{r}$

By induction hypothesis, we know that $\mathbf{r} = \sum_{i=1}^{n} \alpha_i \cdot \mathbf{b}_i + \sum_{j=n+1}^{m} \mathbf{b}_j \in \mathbb{V}$. We consider the following cases:

- If $m \neq n+1$ or $n \neq 0$, then at least one reduction rule from Group E can be applied, hence $(\mathbf{r})\ \mathbf{s} \notin \mathsf{NF}$.
- If $m = n+1$, $n = 0$ and $\alpha = 1$, then $\mathbf{r} = \mathbf{b} \in \mathbb{V}$, but $1 \cdot \mathbf{r} = 1 \cdot \mathbf{b} \to \mathbf{b}$, hence $\alpha \cdot \mathbf{r} \notin \mathsf{NF}$.
- If $m = n+1$, $n = 0$ and $\alpha \neq 1$, then $\mathbf{r} = \mathbf{b} \in \mathbb{V}$ and $\alpha \cdot \mathbf{r} = \alpha \cdot \mathbf{b} \in \mathbb{V}$.

Case $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$

By induction hypothesis, we know that $\mathbf{t}_k = \sum_{i=1}^{n^k} \alpha_i^k \cdot \mathbf{b}_i^k + \sum_{j=n+1}^{m^k} \mathbf{b}_j^k \in \mathbb{V}$, with $k = 1, 2$. We consider the following cases:

- $\exists i, j\ /\ \mathbf{b}_i^1 = \mathbf{b}_j^2$, then at least one reduction rule from Group F can be applied, hence $\mathbf{t}_1 + \mathbf{t}_2 \notin \mathsf{NF}$.
- $\forall i, j\ /\ \mathbf{b}_i^1 \neq \mathbf{b}_j^2$, then by definition of $\mathbb{V}$, $\mathbf{t}_1 + \mathbf{t}_2 \in \mathsf{NF}$. □

5.2 Weight Preservation

As previously discussed, the objective of the system is to be able to model vector spaces. In this context, we know that the basis terms represent base vectors, while general terms represent any vector. From here, it follows that if $\mathbf{v} = \alpha \cdot \mathbf{b}_1 + \beta \cdot \mathbf{b}_2$, then $\mathbf{b}_1$ represents the vector $[1, 0]$, $\mathbf{b}_2$ represents the vector $[0, 1]$, and $\mathbf{v}$ represents the vector $[\alpha, \beta] = \alpha \cdot [1, 0] + \beta \cdot [0, 1]$. Therefore, the weight of $\mathbf{v}$ should be $\alpha + \beta$, since that is effectively the weight of $[\alpha, \beta]$.

This is analogous for types: the unit types represent base vectors (which is why they type basis terms), and the general types represent any vector.

We proceed then to formalise the concept of weight of types and terms. It is worth mentioning that our definition of weight for terms is not complete, in the sense that we define it inductively and only consider the cases we need for our proof: the terms representing applications are not included, since for Weight Preservation we are only considering the weight of value, which means that all the application terms have already been reduced, due to Theorem 5.1.1.

Definition 5.2.1 (Weight of types). *We define the relation $\mathcal{W}(\bullet)$: Type $\to$ Scalar inductively as follows:*

- $\mathcal{W}(U) = 1$.

- $\mathcal{W}(\alpha \cdot T) = \alpha \cdot \mathcal{W}(T)$.
- $\mathcal{W}(T + R) = \mathcal{W}(T) + \mathcal{W}(R)$.

Example 5.2.2. *Consider the type* $\sum_{i=1}^{n} \alpha_i \cdot U_i$*, then*

$$\mathcal{W}\left(\sum_{i=1}^{n} \alpha_i \cdot U_i\right) = \sum_{i=1}^{n} \alpha_i \cdot \mathcal{W}(U_i)$$
$$= \sum_{i=1}^{n} \alpha_i$$

Definition 5.2.3 (Weight of terms)**.** *We define the relation* $\mathcal{W}(\bullet)$: *Term* $\rightarrow$ *Scalar inductively as follows:*

- $\mathcal{W}(\mathbf{b}) = 1$.
- $\mathcal{W}(\alpha \cdot \mathbf{t}) = \alpha \cdot \mathcal{W}(\mathbf{t})$.
- $\mathcal{W}(\mathbf{t} + \mathbf{r}) = \mathcal{W}(\mathbf{t}) + \mathcal{W}(\mathbf{r})$.

Example 5.2.4. *Consider the term* $\sum_{i=1}^{n} \alpha_i \cdot \mathbf{b}_i$*, then*

$$\mathcal{W}\left(\sum_{i=1}^{n} \alpha_i \cdot \mathbf{b}_i\right) = \sum_{i=1}^{n} \alpha_i \cdot \mathcal{W}(\mathbf{b}_i)$$
$$= \sum_{i=1}^{n} \alpha_i$$

Lemma 5.2.5. *If* $T \equiv R$*, then* $\mathcal{W}(T) = \mathcal{W}(R)$.

Proof. We prove the lemma holds for every definition of $\equiv$

Case $1 \cdot T \equiv T$

Trivial case.

Case $\alpha \cdot (\beta \cdot T) \equiv (\alpha \times \beta) \cdot T$

$$\mathcal{W}(\alpha \cdot (\beta \cdot T)) = \alpha \cdot \mathcal{W}(\beta \cdot T) = (\alpha \times \beta) \cdot \mathcal{W}(T) = \mathcal{W}((\alpha \times \beta) \cdot T)$$

Case $\alpha \cdot T + \alpha \cdot R \equiv \alpha \cdot (T + R)$

$$\begin{aligned} \mathcal{W}(\alpha \cdot T + \alpha \cdot R) &= \mathcal{W}(\alpha \cdot T) + \mathcal{W}(\alpha \cdot R) \\ &= \alpha \cdot \mathcal{W}(T) + \alpha \cdot \mathcal{W}(R) = \alpha \cdot (\mathcal{W}(T) + \mathcal{W}(R)) \\ &= \alpha \cdot (\mathcal{W}(T + R)) = \mathcal{W}(\alpha \cdot (T + R)) \end{aligned}$$

Case $\alpha \cdot T + \beta \cdot T \equiv (\alpha + \beta) \cdot T$

$$\begin{aligned} \mathcal{W}(\alpha \cdot T + \beta \cdot T) &= \mathcal{W}(\alpha \cdot T) + \mathcal{W}(\beta \cdot T) = \alpha \cdot \mathcal{W}(T) + \beta \cdot \mathcal{W}(T) \\ &= (\alpha + \beta) \cdot \mathcal{W}(T) = \mathcal{W}((\alpha + \beta) \cdot T) \end{aligned}$$

Case $T + R \equiv R + T$

$$\mathcal{W}(T + R) = \mathcal{W}(T) + \mathcal{W}(R) = \mathcal{W}(R) + \mathcal{W}(T) = \mathcal{W}(T + R)$$

Case $T + (R + S) \equiv (T + R) + S$

$$\begin{aligned}\mathcal{W}(T + (R + S)) &= \mathcal{W}(T) + \mathcal{W}(R + S) = \mathcal{W}(T) + \mathcal{W}(R) + \mathcal{W}(S) \\ &= \mathcal{W}(T + R) + \mathcal{W}(S) = \mathcal{W}((T + R) + S)\end{aligned}$$

□

Lemma 5.2.6. *If* $\vdash \mathbf{v} = \sum_{i=1}^{k} \alpha_i \cdot \mathbf{b}_i + \sum_{i=k+1}^{n} \mathbf{b}_i : T$, *then* $\mathcal{W}(T) \equiv \mathcal{W}(\mathbf{v})$.

Proof. We proceed by induction on n.

Case $n = 1$

There are two possible escenarios:

$k = 1$

In this scenario, consider $\pi = \vdash \alpha_1 \cdot \mathbf{b}_1 : T$. By Lemma 4.1.10, there exist $R_1, \dots, R_m$, $\beta_1, \dots, \beta_m$ such that

- $T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j$.
- $\pi_i = \Gamma \vdash \mathbf{b}_1 : R_j$, with $size(\pi) > size(\pi_j)$, for $j \in \{1, \dots, m\}$.
- $\sum_{i=1}^{m} \beta_j = \alpha_1$.

Then by Lemma 4.1.14, for each $j \in \{1, \dots, m\}$ (we will omit the j index for readability), there exist $U_1, \dots, U_h$, $\sigma_1, \dots, \sigma_h$ such that

- $R \equiv \sum_{k=1}^{h} \sigma_k \cdot U_k$.
- $\Gamma \vdash \mathbf{b}_1 : U_k$, for $k \in \{1, \dots, h\}$.
- $\sum_{k=1}^{h} \sigma_k = 1$.

Then,

$$T \equiv \sum_{j=1}^{m} \beta_j \cdot R_j \equiv \sum_{j=1}^{m} \beta_j \cdot (\sum_{k=1}^{h_j} \sigma_{(j,k)} \cdot U_{(j,k)})$$

Finally, by definition of $\mathcal{W}(\bullet)$, we have

$$\begin{aligned}\mathcal{W}(\mathbf{v}) &= \mathcal{W}\left(\sum_{i=1}^{1} \alpha_i \cdot \mathbf{b_i}\right) = \sum_{i=1}^{1} \alpha_i \cdot \mathcal{W}(\mathbf{b_i}) \\ &= \sum_{i=1}^{1} \alpha_i = \alpha_1 = \sum_{i=1}^{m} \beta_j = \sum_{i=1}^{m} \beta_j \cdot \underbrace{\left(\sum_{k=1}^{h_j} \sigma_{(j,k)}\right)}_{=1} \\ &= \sum_{i=1}^{m} \beta_j \cdot \left(\sum_{k=1}^{h_j} \sigma_{(j,k)} \cdot \mathcal{W}\left(U_{(j,k)}\right)\right) \\ &= \sum_{i=1}^{m} \beta_j \cdot \mathcal{W}\left(\sum_{k=1}^{h_j} \sigma_{(j,k)} \cdot U_{(j,k)}\right) = \mathcal{W}\left(\sum_{i=1}^{m} \beta_j \cdot \left(\sum_{k=1}^{h_j} \sigma_{(j,k)} \cdot U_{(j,k)}\right)\right) \\ &= \mathcal{W}(T)\end{aligned}$$

$k = 0$ ....

In this scenario, consider $\vdash \mathbf{b}_1 : T$. By Lemma 4.1.14, there exist $U_1, \ldots, U_m$, $\beta_1, \ldots, \beta_m$ such that

- $T \equiv \sum_{j=1}^{m} \beta_k \cdot U_j$.
- $\Gamma \vdash \mathbf{b}_1 : U_j$, for $j \in \{1, \ldots, m\}$.
- $\sum_{j=1}^{m} \beta_j = 1$.

Finally, by definition of $\mathcal{W}(\bullet)$, we have

$$\begin{aligned}\mathcal{W}(\mathbf{v}) =& \mathcal{W}(\mathbf{b}_1) = 1 = \sum_{j=1}^{m} \beta_j \\ =& \sum_{j=1}^{m} \beta_j \cdot \mathcal{W}(U_j) = \mathcal{W}\left(\sum_{j=1}^{m} \beta_j \cdot U_j\right) \\ =& \mathcal{W}(T)\end{aligned}$$

.... **Induction step** ....

Consider now that $\vdash \mathbf{v} = \mathbf{v}' + \mathbf{v}'' : T$, where $\mathbf{v}' = \sum_{i=1}^{k} \alpha_i \cdot \mathbf{b}_i + \sum_{j=k+1}^{n} \mathbf{b}_j$ and either $\mathbf{v}'' = \beta \cdot \mathbf{b}$, or $\mathbf{v}'' = \mathbf{b}$. By Lemma 4.1.11, we know there exists R and S such that

- $T \equiv R + S$.
- $\Gamma \vdash \mathbf{v}' : R$.
- $\Gamma \vdash \mathbf{v}'' : S$.

By induction hypothesis, since $\vdash \mathbf{v}' = \sum_{i=1}^{k} \alpha_i \cdot \mathbf{b}_i + \sum_{j=k+1}^{n} \mathbf{b}_j : R$, then $\mathcal{W}(R) = \mathcal{W}(\mathbf{v}')$; and since either $\mathbf{v}'' = \beta \cdot \mathbf{b}$ or $\mathbf{v}'' = \mathbf{b}$, in both cases we know that $\mathcal{W}(S) = \mathcal{W}(\mathbf{v}'')$. Finally, and considering by Lemma 5.2.5 that $\mathcal{W}(T) = \mathcal{W}(R) + \mathcal{W}(S)$, we have

$$\begin{aligned}\mathcal{W}(\mathbf{v}) =& \mathcal{W}(\mathbf{v}' + \mathbf{v}'') \\ =& \mathcal{W}(\mathbf{v}') + \mathcal{W}(\mathbf{v}'') \\ =& \mathcal{W}(R) + \mathcal{W}(S) \\ =& \mathcal{W}(T)\end{aligned}$$

□

Theorem 5.2.7 (Weight Preservation). *If* $\vdash \mathbf{t} : T$ *and* $\mathbf{t} \to^* \mathbf{v}$, *then* $\mathcal{W}(T) = \mathcal{W}(\mathbf{v})$.

Proof. Since $\mathbf{t} \to^* \mathbf{v}$, by Theorem 5.1.1, $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \cdot \mathbf{b}_i + \sum_{j=1}^{m} \mathbf{b}_j$, where $\mathbf{b}_i \neq \mathbf{b}_j$ for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots m\}$. Also, by Theorem 4.2.1, we know then that $\vdash \mathbf{v} : T$. Finally, by Lemma 5.2.6, we know that $\mathcal{W}(T) = \mathcal{W}(\mathbf{v})$. □

Chapter 6

Conclusion

6.1 Summary

We have introduced $\lambda^{\text{vec*}}$ and proved that it satisfies the standard formulation of the Subject Reduction property (Theorem 4.2.1), which guarantees that upon reducing a term, its type will be preserved. It is worth mentioning that in the process of doing so, we faced several problems regarding the changes we needed to make to the original λ^{vec} system. Indeed, one of the first approaches we considered involved keeping most of the typing rules as in the original system, adding subtyping. The main problem with such approach was that, besides making the system more complex, the proofs became unnecessarily complex as well.

In the end, we realized that the property could be satisfied just by modifying the typing rules, which yielded a simpler and more elegant system than the one we first devised. The summary of the changes made to the original system is:

- We added the S rule, that deals with superposition of types of a single term:

$$\frac{\Gamma \vdash \mathbf{t} : T_i \ \forall i \in \{1, \dots, n\}}{\Gamma \vdash \left(\sum_{i=1}^{n} \alpha_i\right) \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot T_i}\, S$$

- We added the 1_E rule, to allow the removal of the scalar if said scalar is equal to 1:

$$\frac{\Gamma \vdash 1 \cdot \mathbf{t} : T}{\Gamma \vdash \mathbf{t} : T}\, 1_E$$

- We relaxed the $\forall$ rules to only predicate over a given summand at a time, instead of all the summands at once:

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i \cdot U_i \quad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n}\, \forall_I \qquad \frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot \forall X.U_n}{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n-1} \alpha_i \cdot U_i + \alpha_n \cdot U_n[A/X]}\, \forall_E$$

- We removed the term $\mathbf{0}$, to avoid introducing typing rules just to handle the rewrite rules associated with it in the Subject Reduction proof.

We were also able to prove Progress (Section 5.1), which allowed us to characterise the terms that cannot be reduced any further. This particular result was very significant since it enabled us to formalise the concept of weight of types and terms, and to prove that terms had the same weight as their types (Section 5.2).

6.2 Future directions

In the following paragraphs, we present some unproven intuitions of properties that we believe can serve as a starting point for future work.

6.2.1 Strong Normalisation

In Section 2.3 we mentioned that one of the properties satisfied by λ^{Vec} was Strong Normalisation. Since the main focus of our revision was to recover the standard formulation of the Subject Reduction property, we did not prove if $\lambda^{\mathrm{Vec}*}$ satisfied the Strong Normalisation property as well. However, we believe that $\lambda^{\mathrm{Vec}*}$ satisfies this property since it could be possible to prove the sequents of $\lambda^{\mathrm{Vec}*}$ and λ^{Vec} are related:

If $\Gamma \vdash \mathbf{t} : T$, then there exists R such that $\Gamma \vdash_{\lambda_{\mathsf{Vec}}} \mathbf{t} : R$.

If that statement is proved, and since the terms in λ^{Vec} are strongly normalising, then it follows that the terms in $\lambda^{\mathrm{Vec}*}$ must be strongly normalising as well.

A.1 Interpretation of typing judgements

A.1.1 The general case

In the general case the calculus can represent infinite-dimensional linear operators such as $\lambda x.x$, $\lambda x.\lambda y.y$, $\lambda x.\lambda f.(f)\,x$,... and their applications. Even for such general terms $\mathbf{t}$, the vectorial type system provides much information about the superposition of basis terms $\sum_i \alpha_i \cdot \mathbf{b}_i$ to which $\mathbf{t}$ is reduced to, as proven in Section 5.1. How much information is brought by the type system in the finitary case is the topic of Section A.1.2.

A.1.2 The finitary case: Expressing matrices and vectors

In what we call the "finitary case", we show how to encode finite-dimensional linear operators, i.e. matrices, together with their applications to vectors, as well as matrix and tensor products. The encoding of 2-dimensional vectors differs from that of λ^{vec}, but the general encodings are the same [3, 6. Interpretation of typing judgements]. We still show all the encodings in this section.

In 2 dimensions

In this section we come back to the motivating example introducing the type system and we show how $\lambda^{\mathrm{vec}*}$ handles the Hadamard gate, and how to encode matrices and vectors.

With an empty typing context, the booleans $\mathbf{true} = \lambda x.\lambda y.x$ and $\mathbf{false} = \lambda x.\lambda y.y$ can be respectively typed with the types $\mathcal{T} = \forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to (\mathcal{Y} \to \mathcal{X})$ and $\mathcal{F} = \forall \mathcal{X}\mathcal{Y}.\mathcal{X} \to (\mathcal{Y} \to \mathcal{Y})$. The superposition has the following type $\vdash \alpha \cdot \mathbf{true} + \beta \cdot \mathbf{false} : \alpha \cdot \mathcal{T} + \beta \cdot \mathcal{F}$. (Note that it can also be typed with $(\alpha + \beta) \cdot \forall \mathcal{X}.\mathcal{X} \to \mathcal{X} \to \mathcal{X}$).

The linear map $\mathbf{U}$ sending $\mathbf{true}$ to $a \cdot \mathbf{true} + b \cdot \mathbf{false}$ and $\mathbf{false}$ to $c \cdot \mathbf{true} + d \cdot \mathbf{false}$, that is

$$\mathbf{true} \mapsto a \cdot \mathbf{true} + b \cdot \mathbf{false},$$
$$\mathbf{false} \mapsto c \cdot \mathbf{true} + d \cdot \mathbf{false}$$

is written as

$$\mathbf{U} = \lambda x.\,\{((x)\ \ [a \cdot \mathbf{true} + b \cdot \mathbf{false}])\ \ [c \cdot \mathbf{true} + d \cdot \mathbf{false}]\}.$$

The following sequent is valid:

$$\vdash \mathbf{U} : \forall \mathbb{X}.((I \to (a \cdot \mathcal{T} + b \cdot \mathcal{F})) \to (I \to (c \cdot \mathcal{T} + d \cdot \mathcal{F})) \to I \to \mathbb{X}) \to \mathbb{X}.$$

This is consistent with the discussion in the introduction: the Hadamard gate is the case $a = b = c = \frac{1}{\sqrt{2}}$ and $d = -\frac{1}{\sqrt{2}}$. One can check that with an empty typing context, $(\mathbf{U})\ \mathbf{true}$ is well typed of type $a \cdot \mathcal{T} + b \cdot \mathcal{F}$, as expected since it is reduced to $a \cdot \mathbf{true} + b \cdot \mathbf{false}$:

$$\begin{aligned}
(\mathbf{U})\ \mathbf{true} &= (\lambda x.\,\{((x)\,[a \cdot \mathbf{true} + b \cdot \mathbf{false}])\,[c \cdot \mathbf{true} + d \cdot \mathbf{false}]\})\ \ (\lambda x.\lambda y.x) \\
&= \lambda x.\,((((x)\ \ (\lambda f.a \cdot \mathbf{true} + b \cdot \mathbf{false}))\ \ (\lambda g.c \cdot \mathbf{true} + d \cdot \mathbf{false}))\ \ (\lambda x.x))\ \ (\lambda x.\lambda y.x) \\
&\to (((\lambda x.\lambda y.x)\ (\lambda f.a \cdot \mathbf{true} + b \cdot \mathbf{false}))\ (\lambda g.c \cdot \mathbf{true} + d \cdot \mathbf{false}))\ (\lambda x.x) \\
&\to ((\lambda y.\lambda f.a \cdot \mathbf{true} + b \cdot \mathbf{false})\ (\lambda g.c \cdot \mathbf{true} + d \cdot \mathbf{false}))\ (\lambda x.x) \\
&\to (\lambda f.a \cdot \mathbf{true} + b \cdot \mathbf{false})\ (\lambda x.x) \\
&\to a \cdot \mathbf{true} + b \cdot \mathbf{false}
\end{aligned}$$

The term $(\mathbf{H})\ \frac{1}{\sqrt{2}} \cdot (\mathbf{true} + \mathbf{false})$ is well-typed of type $\mathcal{T} + 0 \cdot \mathcal{F}$.

$$\begin{aligned}
(\mathbf{H})\ \left(\frac{1}{\sqrt{2}} \cdot (\mathbf{true} + \mathbf{false})\right) &\to^* \left((\mathbf{H})\ \left(\frac{1}{\sqrt{2}} \cdot \mathbf{true}\right)\right) + \left((\mathbf{H})\ \left(\frac{1}{\sqrt{2}} \cdot \mathbf{false}\right)\right) \\
&\to^* \frac{1}{\sqrt{2}} \cdot ((\mathbf{H})\ \mathbf{true}) + \frac{1}{\sqrt{2}} \cdot ((\mathbf{H})\ \mathbf{false}) \\
&\to^* \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} \cdot \mathbf{true} + \frac{1}{\sqrt{2}} \cdot \mathbf{false}\right) + \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} \cdot \mathbf{true} - \frac{1}{\sqrt{2}} \cdot \mathbf{false}\right) \\
&\to^* \frac{1}{2} \cdot \mathbf{true} + \frac{1}{2} \cdot \mathbf{false} + \frac{1}{2} \cdot \mathbf{true} - \frac{1}{2} \cdot \mathbf{false} \\
&\to 1 \cdot \mathbf{true} + 0 \cdot \mathbf{false} \\
&\to \mathbf{true} + 0 \cdot \mathbf{false}
\end{aligned}$$

Since the term is reduced to $\mathbf{true} + 0 \cdot \mathbf{false}$, this is consistent with the subject reduction.

But we can do more than typing 2-dimensional vectors or 2×2-matrices: using the same technique we can encode vectors and matrices of any size.

Vectors in n dimensions

The 2-dimensional space is represented by the span of $\lambda x_1 x_2.x_1$ and $\lambda x_1 x_2.x_2$: the n-dimensional space is simply represented by the span of all the $\lambda x_1 \cdots x_n.x_i$, for $i \in \{1, \ldots, n\}$. As for the two dimensional case where

$$\vdash \alpha_1 \cdot \lambda x_1 x_2.x_1 + \alpha_2 \cdot \lambda x_1 x_2.x_2 : \alpha_1 \cdot \forall \mathcal{X}_1 \mathcal{X}_2.\mathcal{X}_1 + \alpha_2 \cdot \forall \mathcal{X}_1 \mathcal{X}_2.\mathcal{X}_2,$$

an n-dimensional vector is typed with

$$\vdash \sum_{i=1}^{n} \alpha_i \cdot \lambda x_1 \cdots x_n.x_i : \sum_{i=1}^{n} \alpha_i \cdot \forall \mathcal{X}_1 \cdots \mathcal{X}_n.\mathcal{X}_i.$$

We use the notations

$$\mathbf{e}_i^n = \lambda x_1 \cdots x_n.x_i, \qquad \mathbf{E}_i^n = \forall \mathcal{X}_1 \cdots \mathcal{X}_n.\mathcal{X}_i$$

and we write

$$\left[\!\!\left[\begin{matrix} \alpha_1 \\ \vdots \\ \alpha_n \end{matrix} \right]\!\!\right]_n^{\text{term}} = \begin{pmatrix} \alpha_1 \cdot \mathbf{e}_1^n \\ + \\ \cdots \\ + \\ \alpha_n \cdot \mathbf{e}_n^n \end{pmatrix} = \sum_{i=1}^{n} \alpha_i \cdot \mathbf{e}_i^n,$$

$$\left[\!\!\left[\begin{matrix} \alpha_1 \\ \vdots \\ \alpha_n \end{matrix} \right]\!\!\right]_n^{\text{type}} = \begin{pmatrix} \alpha_1 \cdot \mathbf{E}_1^n \\ + \\ \cdots \\ + \\ \alpha_n \cdot \mathbf{E}_n^n \end{pmatrix} = \sum_{i=1}^{n} \alpha_i \cdot \mathbf{E}_i^n.$$

$n \times m$ matrices

Once the representation of vectors is chosen, it is easy to generalise the representation of 2×2 matrices to the $n \times m$ case. Suppose that the matrix U is of the form

$$U = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix},$$

then its representation is

$$[\![U]\!]^{\text{term}}_{n\times m} = \quad \lambda x.\left\{\left(\cdots\left((x)\begin{bmatrix}\alpha_{11}\cdot\mathbf{e}^n_1\\+\\\cdots\\+\\\alpha_{n1}\cdot\mathbf{e}^n_n\end{bmatrix}\right)\cdots\begin{bmatrix}\alpha_{1m}\cdot\mathbf{e}^n_1\\+\\\cdots\\+\\\alpha_{nm}\cdot\mathbf{e}^n_n\end{bmatrix}\right)\right\}$$

and its type is

$$[\![U]\!]^{\text{type}}_{n\times m} = \quad \forall\mathbb{X}.\left(\begin{bmatrix}\alpha_{11}\cdot\mathbf{E}^n_1\\+\\\cdots\\+\\\alpha_{n1}\cdot\mathbf{E}^n_n\end{bmatrix}\to\cdots\to\begin{bmatrix}\alpha_{1m}\cdot\mathbf{E}^n_1\\+\\\cdots\\+\\\alpha_{nm}\cdot\mathbf{E}^n_n\end{bmatrix}\to[\ \mathbb{X}\ ]\right)\to\mathbb{X},$$

that is, an almost direct encoding of the matrix U.

We also use the shortcut notation

$$\mathbf{mat}(\mathbf{t}_1,\ldots,\mathbf{t}_n) = \lambda x.(\ldots((x)\ [\mathbf{t}_1])\ldots)\ [\mathbf{t}_n]$$

A.1.3 Useful constructions

In this section, we describe a few terms representing constructions that will be used later on.

Projections The first useful family of terms are the projections, sending a vector to its i^{th} coordinate:

$$\begin{pmatrix}\alpha_1\\\vdots\\\alpha_i\\\vdots\\\alpha_n\end{pmatrix}\longmapsto\begin{pmatrix}0\\\vdots\\\alpha_i\\\vdots\\0\end{pmatrix}.$$

Using the matrix representation, the term projecting the i^{th} coordinate of a vector of size n is

$$\mathbf{p}^n_i = \mathbf{mat}(\mathbf{0},\cdots,\mathbf{0},\ \overset{i^{\text{th}}\text{ position}}{\mathbf{e}^n_i},\ \mathbf{0},\cdots,\mathbf{0}).$$

We can easily verify that

$$\vdash \mathbf{p}^n_i : \left[\!\!\left[\begin{matrix}0&\cdots&0&\cdots&0\\\vdots&\ddots&&&\vdots\\0&&1&&0\\\vdots&&&\ddots&\vdots\\0&\cdots&0&\cdots&0\end{matrix}\right]\!\!\right]^{\text{type}}_{n\times n}$$

and that

$$(\mathbf{p}^n_k)\left(\sum_{i=1}^n\alpha_i\cdot\mathbf{e}^n_i\right)\longrightarrow^*\alpha_k\cdot\mathbf{e}^n_k.$$

Vectors and diagonal matrices Using the projections defined in the previous section, it is possible to encode the map sending a vector of size n to the corresponding $n \times n$ matrix:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \longmapsto \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$$

with the term

$$\mathbf{diag}^n = \lambda b.\mathbf{mat}((\mathbf{p}_1^n)\,\{b\}, \ldots, (\mathbf{p}_n^n)\,\{b\})$$

of type

$$\vdash \mathbf{diag}^n : \left[\left[\!\!\left[\begin{matrix} \alpha_1 \\ \vdots \\ \alpha_n \end{matrix} \right]\!\!\right]_n^{\text{type}} \right] \to \left[\!\!\left[\begin{matrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{matrix} \right]\!\!\right]_{n\times n}^{\text{type}}.$$

It is easy to check that

$$(\mathbf{diag}^n)\,\left[\sum_{i=1}^n \alpha_i \cdot \mathbf{e}_i^n\right] \longmapsto^* \mathbf{mat}(\alpha_1 \cdot \mathbf{e}_1^n, \ldots, \alpha_n \cdot \mathbf{e}_n^n)$$

Extracting a column vector out of a matrix Another construction that is worth exhibiting is the operation

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \longmapsto \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{mi} \end{pmatrix}.$$

It is simply defined by multiplying the input matrix with the i^{th} base column vector:

$$\mathbf{col}_i^n = \lambda x.(x)\,\mathbf{e}_i^n$$

and one can easily check that this term has type

$$\vdash \mathbf{col}_i^n : \left[\!\!\left[\begin{matrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{matrix} \right]\!\!\right]_{m\times n}^{\text{type}} \to \left[\!\!\left[\begin{matrix} \alpha_{1i} \\ \vdots \\ \alpha_{mi} \end{matrix} \right]\!\!\right]_m^{\text{type}}.$$

Note that the same term $\mathbf{col}_i^n$ can be typed with several values of m.

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